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A New Mean-Risk Quality Portfolio Optimization Model

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Abstract

This paper introduces a new mean-risk quality portfolio optimization model, grounded in uncertainty theory. It assumes that the return of assets is an uncertain variable and utilizes data estimated by experts to test the model. The approach begins by using the expected value of return and risk quality to represent the return and risk of the portfolio respectively. A new risk measure, termed "risk quality," is introduced, leading to the establishment of the mean-risk quality portfolio optimization model. Next, to more closely align with real financial markets, the paper integrates constraints reflecting realistic financial market characteristics. These include transaction costs, financing constraints, and threshold constraints. By doing so, it establishes a mean-risk quality model with minimized risk, providing a more robust portfolio optimization framework. Finally, both the basic mean-risk-quality model and the version incorporating realistic constraints are empirically analyzed using genetic algorithms. Sensitivity tests are conducted on the robust portfolio model with minimized risk to further validate its effectiveness.

Keywords: risk-quality, portfolio optimization model, reality constraints, genetic algorithms

1. Introduction

The development of the stock market originated in the Amsterdam Stock Exchange in the 17th century, where people sought to increase wealth through investment in business ventures. With the industrialization of the 19th and 20th centuries, the stock market's importance grew. Today, it stands as a crucial hub for business and individual investment and finance. Many investors are attracted to the high returns, but these are often accompanied by high risks. Failure is imminent for investors who cannot balance these elements. Therefore, the issue of portfolio optimization, aimed at maintaining stock market prosperity by balancing risk and return, is a challenge that urgently needs addressing.

This paper explores the portfolio optimization problem using uncertainty theory, constructing a suitable estimation method for downside risk. While most scholars have focused on portfolio risk estimation methods grounded in statistics and probability theory (Siew and Lam 2021; Khodamoradi 2023; Indarwati 2021), others have proposed risk estimation based on fuzzy mathematical theory (Zhang et al. 2017; Yang et al. 2022; Deliktas and Ustun 2022). However, the determination of the affiliation function in fuzzy mathematics is subjective, and there is a lack of objective criteria for this function. This paper synthesizes these limitations and explores portfolio optimization using uncertainty theory. Some scholars use uncertainty theory to construct a risk index for portfolio risk estimation. They treat return below a certain level as a loss and the expected uncertainty measure of this loss as the risk estimate. However, this approach may not reflect the portfolio's complete risk, ignoring how different loss levels may bring varying risk degrees.

Based on the above analysis, the research seeks to answer these questions:

- Can we design a portfolio risk estimation method based on uncertainty theory?
- How can we quantify the real constraint characteristics in the financial market?
- Is it possible to apply the model to the real stock market?

The objective is to offer an appropriate portfolio optimization method for balancing risk and return. This paper introduces "risk quality," which combines portfolio return volatility with an uncertainty measure of losses to estimate downside risk. A mean-risk quality portfolio optimization model is developed, integrating risk quality and expected return. Additionally, real constraints in financial markets, such as threshold constraints and transaction cost constraints, are quantified. The research contributions include:

- Merging the limitations of probability theory and fuzzy mathematics in previous studies and choosing

uncertainty theory to address the portfolio optimization problem.
- Creating a new downside risk estimation method, which combines volatility and uncertainty measures of portfolio returns.
- Developing mean-risk-quality portfolio optimization models with realistic constraints, testing the risk quality's ability to estimate downside risk, and assessing model validity.

The remainder of this paper is organized into distinct sections. Section 2 provides a summary of the current state of research on the subject. In Section 3, we introduce the research methodology and describe the data sources used in our study. Section 4 then presents the empirical results, along with a detailed analysis. Section 5 delves further into the model's applicability and usefulness. Finally, Section 6 offers a comprehensive summary of the entire paper and outlines future directions in this area.

2. Literature Review

2.1 Research on the Concepts of Portfolio Optimization Problems

Portfolio optimization involves selecting an optimal portfolio in the securities market, enabling investors to maximize returns while minimizing risk. Markowitz (1952) first characterized portfolio return and risk in terms of the expected value and variance of stocks, thereby proposing the classical mean-variance model. This marked the inception of portfolio optimization theory.

Risk control is pivotal in portfolio optimization. While variance measures volatility both below and above the sample mean, it's criticized for equating portfolio return and risk as an unbalanced measure. Investors are primarily concerned with downside risk, and Markowitz's use of downside semivariance offers a more nuanced risk measurement. Many scholars have further advanced portfolio optimization by devising various risk estimation indicators.

2.2 Research on Portfolio Risk

Research on portfolio risk can be categorized into two main areas:

1) Statistical Theory-Based Approach. This includes methods such as mean and lower half variance estimation (Mao 1970), mean-absolute deviation model (Konno and Yamazaki 1991), and other variance-rooted approaches. Scholars have also explored alternative risk estimation methods like lower semi-variance or absolute deviation (Meng 2021; Siew 2020; Zhang 2018).
2) Probabilistic Theory-Based Approach. Centered around Value at Risk (VaR) and Conditional Value at Risk (CVaR), this approach has been both lauded and questioned. For instance, VaR's inability to capture risk spillovers was exposed during the 2008 financial crisis, leading to the development and adoption of CVaR as a risk measurement tool (Salahi 2023; Khodamoradi 2023).

However, both approaches have limitations, particularly when considering the human influence on financial markets and unpredictable events like financial crises. This has led to the exploration of uncertainty theory as a solution (Liu 2007, 2015).

Recent studies have advanced the portfolio problem in the field of uncertainty by building new optimization models (Shen 2023; Guo 2022; Ramedani 2022). For instance, Huang's risk index (2012) centered on uncertainty theory offers an innovative estimation method for portfolio optimization. The World Bank's application of the risk index (2021) for dam risk screening and the continuous improvement of uncertainty theory further demonstrates its relevance in the field.

2.3 Research on the Application of Portfolio Optimization Models

The existing research has significantly advanced portfolio optimization, but practical application in financial markets remains challenging due to various realistic constraints such as transaction costs, financing constraints, and threshold constraints. (1) Transaction Costs: Primbs (2019) identified transaction costs as essential considerations in portfolio optimization. Recent studies have further quantified these costs for different financial markets (Guo 2023; Pitera 2023). (2) Financing Constraints and Threshold Constraints: Self-financing, upper and lower limits must be considered to realistically predict future stock returns. Research in this direction includes multi-objective portfolio models with transaction costs and financing constraints (Maisa 2022; M. S. Al-Nator 2020). (3) Other Realistic Constraints: Scholars have also explored constraints like background risk, mental accounts, and inflation (Huang 2022; Hübner 2022; Vukovi 2022; Liu 2023). These realistic constraints do not exist in isolation, necessitating models that account for the interplay between them (Jiang and Huang 2023; Markowitz and Statman 2010).

While existing literature on portfolio optimization has yielded valuable insights, certain limitations persist. Current research often falls into either probability-based or fuzzy mathematics-based approaches, each with its restrictions. There is a relative dearth of research on portfolio optimization under uncertainty theory, and a notable gap exists in applying these models to real financial markets.

This paper aims to creatively construct a downside risk estimation method, quantify realistic constraints under uncertainty theory, and apply the model to actual financial markets using expert-estimated data. By utilizing genetic algorithms for empirical analysis, we seek to provide investors with a securities investment model more attuned to the real financial market.

3. Method

3.1 Symbolic Descriptions and Assumptions

This section outlines the assumptions and provides notational descriptions relevant to the study. Given the inherent uncertainty in real financial markets and the constraints faced in portfolio optimization due to realistic characteristics, the following assumptions are made:

Assumpation 1. Investors act rationally, seeking to maximize return while minimizing risk in the financial market.

Assumpation 2. Investors vary in their degrees of risk aversion.

Assumpation 3. In the financial market, different stocks exhibit varied rates of return and risk profiles, often correlating higher risk with greater return.

Assumpation 4. The financial market is sufficiently advanced, offering a wide selection of stocks with ample liquidity.

Table 1 elucidates the symbols commonly used throughout this paper.

Table 1. Symbols descriptions

Symbols	Meanings
x_i	Percentage of the total capital invested in equities.
x_f	The percentage of total invested funds allocated to risk-free assets.
r_i	The uncertainty in the rate of return on stock i.
c	The transaction cost per unit of stock.
x_i^t	The proportion of investment in equity during period t.
σ_i	Standard deviation of stock.
l^b	The lower bound of the investment ratio of the stock.
u^b	The upper bound of the investment ratio of stock.
r^f	Return on risk-free assets.

3.2 Uncertainty theory

Uncertainty theory, first proposed by Liu (2007), differs from probability theory in that it includes human-controlled factors. It has seen extensive application in a variety of social issues through its continual development. One prominent application lies in the stock market, where the theory can address portfolio selection challenges. Unlike standard probabilistic assumptions, returns on stocks in the market are often assumed to follow distributions characterized by uncertainty, such as normal or linear uncertainty distributions. The following text provides a concise overview of uncertainty theory.

Definition 1. Let Γ be a nonempty set, and L a δ -algebra over Γ. Each element $\Lambda \sim L$ is called an event. A set function $M(\Lambda)$ is called an uncertain measure if it satisfies the following three axioms:

(i) (Normality) $M(\Lambda) = 1$.

(ii) (Self-duality) $M(\Lambda) + M(\Lambda^c) = 1$.

(iii) (Countable subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$M\{\cup_{i=1}^{\infty} \Lambda_i\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

The triplet (Γ, L, M) is called an uncertainty space. That can be proven (Liu 2010) that any uncertain measure M is increasing. That is, for any events $\Lambda_1 \in \Lambda_2$, we have

$$M\{\Lambda_1\} \leq M\{\Lambda_2\}.$$

In order to define product uncertain measure, Liu (2007) proposed the fourth axiom as follows:

(iv) (Product measure) Let(Γ_k, L_k, M_k) be uncertainty spaces for $k = 1, 2, \ldots, n$.

The product uncertain measure is

$$M = M_1 \wedge M_2 \wedge \ldots \wedge M_n.$$

Definition 2. An uncertain variable is a measurable function$\xi$from an uncertainty space(Γ, L, M) to the set of real numbers, i.e., for any Borel set of B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 3. The uncertainty distribution $\Phi: \Re \to [0,1]$ of an uncertain variable ξ is defined by

$$\Phi(t) = M\{\xi \leq t\}.$$

The value obtained from the distribution of the uncertainty function is termed the uncertainty measure, which is also referred to as credibility. For instance, a variable that possesses the following normal uncertainty distribution is referred to as a normal uncertain variable:

$$\phi(r) = \frac{1}{(1+exp(\pi(e-r)/(\sqrt{3}\delta))}, t \in \Re, \quad (1)$$

where e and σ are real numbers and$\delta > 0$. For convenience, it is denoted in the paper by$\xi \sim \mathcal{N}(e, \sigma)$.

In probability theory, the distribution function of a random variable is determined by its probability density function. Unlike probability theory, uncertainty theory does not provide a direct functional expression for an uncertain variable. However, it is possible to derive this expression by understanding the meaning of the uncertain distribution function and its specific mathematical formula. In this study, we determine the functional expression for a 'normal too uncertain variable' by combining the expression of the normal uncertain distribution function with the practical significance of the distribution function within uncertainty theory.

$$\varphi(r) = -\frac{\pi\, exp(\pi(e-r)/(\sqrt{3}\delta)}{\sqrt{3}\delta(1+exp(\pi(e-r)/(\sqrt{3}\delta))^2}, r \in (a, b). \quad (2)$$

Where,(a, b) is denoted as the domain of definition of the function.

It is vital to recognize that within uncertainty theory, an uncertain variable possesses only a corresponding uncertain distribution function. This function is inferred from the actual meaning of the uncertain distribution function and the specific expression that describes it. In particular, it conveys the magnitude of the uncertainty measure associated with the uncertain variable.

Furthermore, employing an uncertainty measure to gauge the fluctuations in securities return addresses a challenge in financial markets. Specifically, the returns on financial assets in real markets do not follow a random distribution. The essence of this approach is that the uncertainty measure accounts for human influences. This allows uncertainty theory to provide a more comprehensive and systematic estimation of financial market behavior than traditional probability theory.

This paper, grounded in uncertainty theory, presents a case to illustrate the distinctions between uncertainty measures and probabilities. Consider the example of tossing a coin and selecting a positive or negative outcome. If the sample size is large enough, and our choice is fixed, the probability of winning approaches 0.5. Conversely, if we invest in a stock and choose between a rising and falling stock price for that day, the probability of winning might become zero under a large enough sample size. This is because if the company encounters an uncertain event in the future that impacts its core technology or assets, then even if the choice is repeated 10,000 times, a price decline is inevitable.

The underlying reason for this outcome is the marked difference between the probability distribution used for stock selection and the real-world circumstances. When the probability distribution of an event deviates from reality, it is unsuitable for conveying the nature of the event. In such cases, treating stock returns as uncertain variables is more appropriate. Therefore, this paper posits that it is feasible to utilize a model, denoted as $\varphi(r)$, to further estimate stock and portfolio risk, assuming that stock returns follow a normal uncertainty distribution.

3.3 Establishment of the Risk Quality

In the framework of uncertainty theory, we present a novel method for estimating risk, which we term "risk quality." This paper defines the rate of return falling below an investor's basic target return as a "loss." Risk quality, then, is the methodology employed to calculate the potential for this loss. It encompasses a unique combination of an uncertain measure of loss, coupled with the volatility of stock returns, to evaluate the risk beneath the investor's fundamental return target.

With the assumption that all stock returns are uncertain variables that follow a normal uncertainty distribution and are independently and identically distributed, this paper synergizes the magnitude of volatility of losses using the uncertainty distribution function. As a result, the derived value encapsulates not only the importance of credibility but also the implications of risk.

Definition 3. Assuming that the return of a stock obeys a certain uncertainty distribution, then its uncertainty measure function is$\psi(r)$ and the uncertainty distribution is$\Psi(r)$. Then, the risk quality can be designed in the following form:

$$RQ = \int_a^{r_f}(r_f - r)\psi(r)dr\,, \quad r \in (a, r_f). \tag{3}$$

Where, a is the lower bound of the independent variable and$r_f$is the upper bound of the independent variable.

In this paper, we posit that stock returns follow a normal distribution of uncertainty and are independently and identically distributed. Analysis of model (3) reveals a convergence problem in the risk-quality quilt function when $a = -\infty$, occurring before the integral function is solved. Assuming the financial market permits short selling, the lower limit of risk quality can extend to negative infinity, where the risk quality becomes convergent.

A proof is provided in Appendix 1.

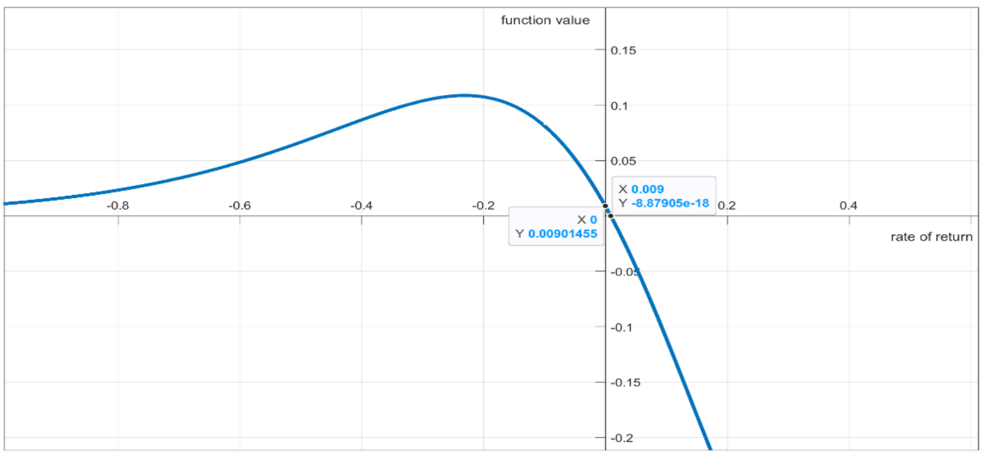

Figure 1. Function image of the risk-quality quilt product function

Therefore, the integral function of risk quality is convergent at$r \to -\infty$, which means that the risk quality is convergent at$r \to -\infty$. As Figure 1 shows the function image of the integrand function of risk quality assuming that stock returns obey a normal uncertain distribution, it can be seen from the figure that when$r \to -\infty$, the value of the integrand function converges to 0 infinitely; and when$r \to +\infty$, the value of the integrand function converges to negative infinity. What needs to be noticed is that when$r = 0.009$, the value of the integrand function is infinitely close to 0.

Obviously, the formula above is only a computational expression for estimating a stock, then the risk quality of a portfolio should be written as

$$RQ = \sum_{i=1}^{n} x_i(a - r_f)\phi_i(a) + \sum_{i=1}^{n} \frac{\sqrt{3}\delta_i x_i}{\pi} ln(\frac{1-\chi}{1-\beta})\,, \tag{4}$$

where,$\chi = \frac{exp[\pi(a - \sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i\sigma_i)]}{1 - exp[\pi(a - \sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i\sigma_i)]}, \beta = \frac{exp[\pi(r_f - \sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i\sigma_i)]}{1 - exp[\pi(r_f - \sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i\sigma_i)]}$.

The proof 2 is in the Appendix section.

3.4 Mean-Risk Quality Model

This section begins with a mean-risk quality portfolio optimization model with a risk capacity constraint. The model that maximizes returns for a given risk capacity is shown as follows:

$$\begin{cases} \max \ E(\sum_{i=1}^{n} x_i r_i) \\ subject\ to: \\ RQ(\sum_{i=1}^{n} x_i\, r_i) \leq M, \\ \sum_{i=1}^{n} x_i = 1. \end{cases} \tag{5}$$

where M is the risk capacity constraint on the quality of risk in the portfolio.

In this paper, we present a specific and clear equivalence form to facilitate the calculation of the model. If the returns of stocks within a portfolio follow a normal uncertain distribution, we can design a portfolio model to estimate both stock returns and risks. This estimation is based on mean and risk quality for the returns and risks, respectively.

$$\begin{cases} \max \ \sum_{i=1}^{n} x_i e_i \\ subject\ to: \\ RQ \leq M, \\ \sum_{i=1}^{n} x_i = 1, \\ RQ = \sum_{i=1}^{n} x_i (a - r_f)\phi_i(a) + \sum_{i=1}^{n} \frac{\sqrt{3}\delta_i x_i}{\pi} ln(\frac{1-\chi}{1-\beta}). \end{cases} \tag{6}$$

Where, $\chi = \frac{exp[\pi(a-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}{1-exp[\pi(a-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}, \beta = \frac{exp[\pi(r_f-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}{1-exp[\pi(r_f-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}$.

Financial markets are inherently complex and chaotic. While model (14) accounts for constraints related to portfolio risk and return pursuit, it overlooks the practical constraints present in actual financial markets, such as transaction costs, borrowing limitations, and threshold constraints. Ignoring these realistic aspects can significantly weaken the model's applicability in real financial markets. Therefore, this section focuses on quantifying these realistic constraints and integrating them into the mean-risk quality model.

Transaction costs are unavoidable for investors to trade in the stock market. In this paper, we only discuss the cost of fees incurred in trading stocks, assuming that the cost of each transaction satisfies the v-shaped function, while the proportional change in the trading of the ith stock in period t can be expressed as $|x_i^t - x_i^{t-i}|$,Then, the transaction cost can be expressed as

$$V = c\sum_{i=1}^{n} |x_i^t - x_i^{t-1}|, \tag{7}$$

The portfolio return g after adding the transaction cost constraint should be adjusted to:

$$g = \sum_{i=1}^{n} x_i e_i - c\sum_{i=1}^{n} |x_i^t - x_i^{t-1}|, \tag{8}$$

The financing constraint refers to the disparity between the interest rate at which individual investors, as opposed to large and well-known firms or central banks, can invest in risk-free assets, compared to the rate at which they can borrow funds. Typically, the interest rate on borrowed funds for these investors exceeds the rate of return on risk-free investments. This distinction is significant and must be considered within the realistic constraints of this paper. Based on the characteristics of the risk-free rate in real financial markets, this paper expresses the risk-free rate in the following form:

$$r_f = \begin{cases} r_f^b, x_f \leq 0, \\ r_f^l, x_f > 0, \end{cases} \tag{9}$$

Where, x_f is the proportion of the portfolio in which investors invest their funds in risk-free assets, r_f^b is the rate of return on investing in risk-free assets, and r_f^b is the interest rate at which investors borrow funds. Then, at this point the portfolio return g containing the transaction cost constraint and the borrowing constraint should be adjusted to：

$$g = \sum_{i=1}^{n} x_i e_i + x_f r_f - c\sum_{i=1}^{n} |x_i^t - x_i^{t-1}|, \tag{10}$$

Threshold Constraint refers to a limitation imposed on the proportion of each stock in a portfolio that must be invested within specified upper and lower bounds. This constraint not only promotes portfolio diversification and manages risk but also allows for flexibility. By altering the range interval or eliminating the short selling limit, the constraint can be tailored to specific needs.

The constraints mentioned above are classified as fundamental reality constraints. The endogeneity among them is weakly correlated; therefore, their inclusion in the same model doesn't lead to substantial endogeneity-induced bias. In this paper, we introduce the risk index (Huang et al., 2012), and innovate by establishing a mean-risk index model that incorporates display constraints. Its formula is expressed in the following form:

$$\begin{cases} \min \mathrm{RI}(\sum_{i=1}^{n}(x_i r_i) + x_f r_f) \\ \quad subject\ to: \\ \mathrm{E}\,[\sum_{i=1}^{n}(x_i r_i) + x_f r_f - c\sum_{i=1}^{n}|x_i^t - x_i^{t-1}|] \geq G, \\ \sum_{i=1}^{n} x_i + x_f = 1, \\ \quad l_i \leq x_i \leq u_i, i = 1,2,\dots,n. \end{cases} \tag{11}$$

At the same time, its simplified form is shown in the following form:

$$\begin{cases} min\ -ln(1-\beta)(\sum_{i=1}^{n}\frac{\sqrt{3}\delta_i x_i}{\pi}) \\ \quad subject\ to: \\ \sum_{i=1}^{n}(x_i - x_{Ei})E[\xi_i] + x_f r_f - c\sum_{i=1}^{n}|x_i^t - x_i^{t-1}| = G, \\ \quad \sum_{i=1}^{n} x_i + x_f = 1, \\ \quad l_i \leq x_i \leq u_i, i = 1,2,\dots,n. \end{cases} \tag{12}$$

where, $\beta = \frac{exp[\pi(r_f - \sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \delta_i)]}{1+exp[\pi(r_f - \sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \delta_i)]}$.

This paper integrates quantitative aspects of transaction costs, financing constraints, and threshold limitations into a mean-risk quality portfolio optimization model. Initially, the model is formulated to incorporate a risk capacity constraint while maximizing portfolio returns. It takes the following form:

$$\begin{cases} \max\ \mathrm{E}\,[\sum_{i=1}^{n}(x_i r_i) + x_f r_f - c\sum_{i=1}^{n}|x_i^t - x_i^{t-1}|] \\ \quad subject\ to: \\ \quad RQ[\sum_{i=1}^{n}(x_i r_i) + x_f r_f] \leq M, \\ \quad \sum_{i=1}^{n} x_i + x_f = 1, \\ \quad l_i \leq x_i \leq u_i, i = 1,2,\dots,n. \end{cases} \tag{13}$$

To validate the mean-risk quality model, this paper juxtaposes it with the mean-risk index model, examining them under realistic constraints. We analyze the capability of risk quality in estimating risk and the efficacy of the mean-risk quality model by contrasting the lower semi-standard deviation differences in portfolios chosen by both models. Subsequently, this section introduces the robust portfolio selection model designed to minimize risk, articulated as follows:

$$\begin{cases} \min\ RQ[\sum_{i=1}^{n}(x_i r_i) + x_f r_f] \\ \quad subject\ to: \\ \mathrm{E}\,[\sum_{i=1}^{n}(x_i r_i) + x_f r_f - c\sum_{i=1}^{n}|x_i^t - x_i^{t-1}|] \geq G, \\ \sum_{i=1}^{n} x_i + x_f = 1, \\ \quad l_i \leq x_i \leq u_i, i = 1,2,\dots,n. \end{cases} \tag{14}$$

To facilitate arithmetic, we simplify the model and give the equivalent form as follows:

$$\begin{cases} min\ \sum_{i=1}^{n} x_i(a - r_f)\phi_i(a) + \sum_{i=1}^{n}(\frac{\sqrt{3}\delta_i x_i}{\pi} ln(\frac{1-\chi}{1-\beta})) \\ \quad subject\ to: \\ \sum_{i=1}^{n}(x_i e_i + x_f r_f - c\sum_{i=1}^{n}|x_i^t - x_i^{t-1}|) \geq G, \\ \quad \sum_{i=1}^{n} x_i + x_f = 1, \\ \quad l_i \leq x_i \leq u_i, i = 1,2,\dots,n. \end{cases}$$

(15)

where, $\chi = \frac{exp[\pi(a-x_f r_f-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}{1-exp[\pi(a-x_f r_f-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}, \beta = \frac{exp[\pi(r_f-x_f r_f-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \delta_i)]}{1+exp[\pi(r_f-x_f r_f-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \delta_i)]}$.

3.5 Data Sources

In this paper, we analyze data from 12 stocks on the China Shanghai Stock Exchange, spanning the period from January 1, 2009, to March 7, 2023. These data are used as the basis for expert analysis and to test the model. We assume that the stock returns follow a normal distribution that is uncertain, and that they satisfy the conditions of independent and identically distributed returns. The estimated returns and standard deviations for the 12 SSE stocks, as determined by experts, are provided in Table 2. Meanwhile, the lower semi-standard deviation for each stock is detailed in Table 3.

Table 2. Uncertain Annual Returns for 12 Stocks in the Chinese Stock Market in 2023

N	Stock code	Returns	N	Stock code	returns
1	601288	$\mathcal{N}(0.0639,0.1879)$	7	601088	$\mathcal{N}(0.1621,0.4181)$
2	601398	$\mathcal{N}(0.0783,0.1993)$	8	601919	$\mathcal{N}(0.1733,0.4324)$
3	601628	$\mathcal{N}(0.1141,0.3406)$	9	600104	$\mathcal{N}(0.1747,0.4369)$
4	600030	$\mathcal{N}(0.1294,0.3541)$	10	600893	$\mathcal{N}(0.2455,0.4563)$
5	601601	$\mathcal{N}(0.1396,0.3671)$	11	600588	$\mathcal{N}(0.2639,0.4702)$
6	600010	$\mathcal{N}(0.1453,0.4100)$	12	603288	$\mathcal{N}(0.2989,0.4738)$

Note: The annual return of the stock is derived using the daily return of 230 trading days.

Table 3. Lower half standard deviation of 12 stocks in the Chinese stock market in 2023

N	lower semi-standard deviation	N	lower semi-standard deviation
1	0.0074	7	0.0127
2	0.0078	8	0.0166
3	0.0125	9	0.0130
4	0.0142	10	0.0178
5	0.0130	11	0.0183
6	0.0157	12	0.0134

Note: The lower bias standard deviation is estimated by experts based on historical data.

3.6 Determination of Genetic Algorithm Parameters

The genetic algorithm serves as a method to find the optimal solution by simulating the natural evolutionary process. This process involves continuous selection of the best solutions through constant generation of random numbers to test a fitness function. While the genetic algorithm is relatively straightforward to implement and compute, improper parameter setting may cause it to become trapped in a local optimum, making it challenging to achieve the global optimal solution of the model. Users of the genetic algorithm should make efforts to mitigate these shortcomings.

In this paper, we work to obtain the global optimal solution more effectively when actually executing the model. We do this by identifying a set of parameters and continuously adjusting them, making it easier to achieve the global optimum of the model.

Before the data operation, on the basis of the actual investment situation in the financial market, we determine the parameters of the portfolio selection model: the rate of return on risk-free assets$r_f^l = 0.09$, ; the interest rate at which the investor borrows funds$r_f^b = 0.017$,; the unit transaction cost$c = 0.003$,; the necessary rate of return required by the investor$g = 0.25$,; the percentage of investment in any stock of the portfolio in the previous period$x_i^t = 1/12$,; and also the minimum rate of return for the investor under the condition that short selling is not allowed$a = -1$. In this paper, we choose A robust mean-risk quality portfolio optimization model with realistic constraints for data experiments, then the model (15) after incorporating the actual model parameters can be rewritten as:

$$\begin{cases} min \ \sum_{i=1}^{n} x_i(-1-0.009)\phi_i(-1) + \sum_{i=1}^{n} \frac{\sqrt{3}\delta_i x_i}{\pi} ln(\frac{1-\chi}{1-\beta}) \\ \quad subject\ to: \\ \quad \sum_{i=1}^{n} x_i E[r_i] + x_f r_f - 0.003 \sum_{i=1}^{n} |x_i^t - x_i^{t-1}| \geq 0.25, \\ \quad \sum_{i=1}^{n} x_i + x_f = 1, \\ \quad 0 \leq x_i \leq 0.3;, i = 1,2,\dots,n, \\ \quad -0.2 \leq x_f \leq 0.3. \end{cases} \tag{16}$$

Where, $\chi = \frac{exp[\pi(-1-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}{1-exp[\pi(-1-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}, \beta = \frac{exp[\pi(0.009-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \delta_i)]}{1+exp[\pi(0.009-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \delta_i)]}$,

$$r_f = \begin{cases} 0.017, x_f \leq 0, \\ 0.009, x_f > 0, \end{cases}$$

We ran Matlab 2020a to solve model (16) using the genetic algorithm and obtained the results presented in Figure 2. We can interpret the image as follows:

The horizontal axis of the figure represents five groups of parameters, each with different crossover and variation probabilities. For instance, the first group consists of parameters (1, 2, 3, 4), with crossover and variation probabilities of 0.5 and 0.8, respectively.

Each group is further divided into four combinations, sharing the same crossover and variation probabilities but with differing population sizes. These are indicated from left to right as 20, 50, 100, and 200, respectively.

Analyzing these data results, we can draw the following conclusions:

•When other parameters are held constant, the population size does not determine whether the operation results converge to the global optimal solution.

•With a fixed population size, the operation results are more likely to converge to the global optimal solution when both crossover and variation probabilities are set to 0.5 and 0.8, respectively.

In solving the portfolio optimization model presented in this paper, we modify the genetic algorithm parameters to facilitate obtaining the global optimal solution. The effect of population size on the model results does not display a consistent trend (where $g_c = 0.5$、$g_m = 0.8$). By varying the population size in each model operation across 10 data sets, we can determine the optimal result and regard it as the global optimum solution.

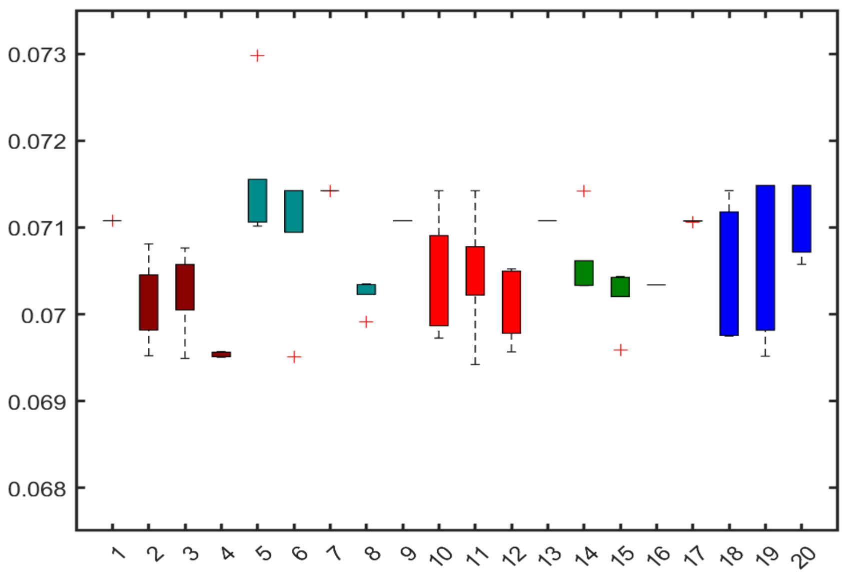

Figure 2. Operational performance of different parameter combinations in the genetic algorithm

4. Results

4.1 Analysis of The Arithmetic Results of The Mean-Risk Quality Portfolio Optimization Model

In this section, we demonstrate the capacity of risk quality to gauge the downside risk of a portfolio, along with the validity of the mean-risk quality model, by analyzing the arithmetic results from various models.

First, by examining the investment landscape in actual financial markets, this paper identifies real values for specific model parameters. These include: $r_f^l = 0.09,$) the rate of return on risk-free assets; $r_f^b = 0.017,$) the unit transaction cost; $c = 0.003$) the interest rate at which the investor borrows funds; and $a = -1$) the minimum rate of return required by the investor, given that short selling is not permitted.

Table 4 presents the arithmetic results of Model (6), assuming that the investor demands a required rate of return, denoted by $r_f = 0.009$. Analysis of these results leads to the following conclusions:

- An increase in risk capacity within the model leads to a corresponding increase in the portfolio's return. This relationship aligns with the widely accepted notion that higher risk often yields greater returns.
- The findings also highlight the efficacy of risk quality in estimating downside risk.

4.2 Analysis of Data Results for Mean-Risk Quality Portfolio Optimization Models with Realistic Constraints

In this section, we present the results of model operations with threshold constraints, as depicted in Figure 3. From the analysis, we can draw two main conclusions:

- The Impact of Threshold Constraint: The model with the threshold constraint exhibits the same characteristics as model (6). As the risk capacity of the model increases, the portfolio return also grows.
- Risk Capacity and Return: The inclusion of the threshold constraint in the model raises the maximum value of risk capacity that might affect changes in the portfolio return. Notably, among the 12 examined stocks, the one with the highest return does not necessarily carry the maximum risk.

To elucidate these findings, we refer to the performance data of stocks with serial numbers 11 and 12. The return and risk quality for stock number 11 are represented as $g = 0.2639, RQ(r_{11}) = 0.0728$, and for stock number 12 as $g = 0.2989, RQ(r_{12}) = 0.0657$. These values reflect common situations in financial markets where stocks may have high returns and low variance.

Estimation of Downside Risk: The estimation of downside risk for stocks 11 and 12, according to risk quality, illustrates this phenomenon. Consequently, using risk quality to assess the risk of stocks aids in the selection of premium stocks, which offer both higher returns and reduced downside risk in the stock market.

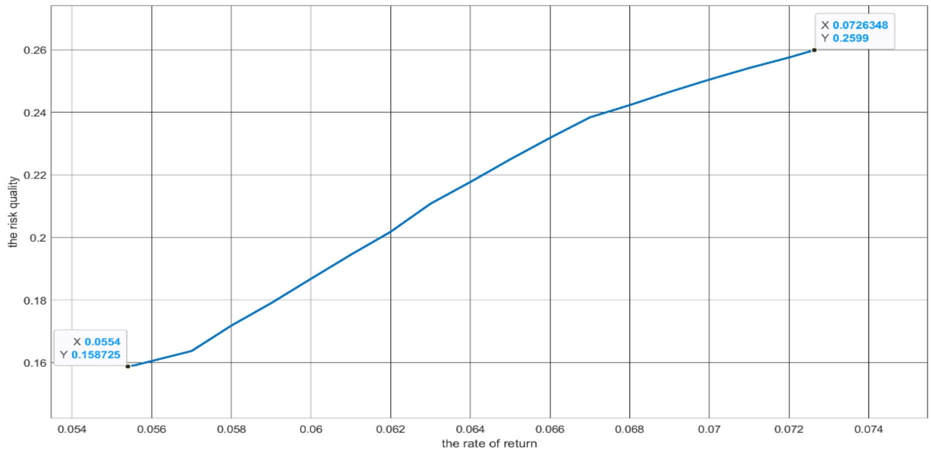

Figure 3. Operational results of the mean-risk quality model with threshold constraints

Figure 4 presents the results for the mean-risk quality model, accounting for threshold constraints, financing constraints, and transaction cost constraints. Analyzing the data, we find that:

- An increase in the model's risk capacity corresponds to a rise in the portfolio's return.
- Compared to model (6), model (13) exhibits a broader range of upper and lower bounds for both risk and return after the inclusion of threshold, financing, and transaction cost constraints.

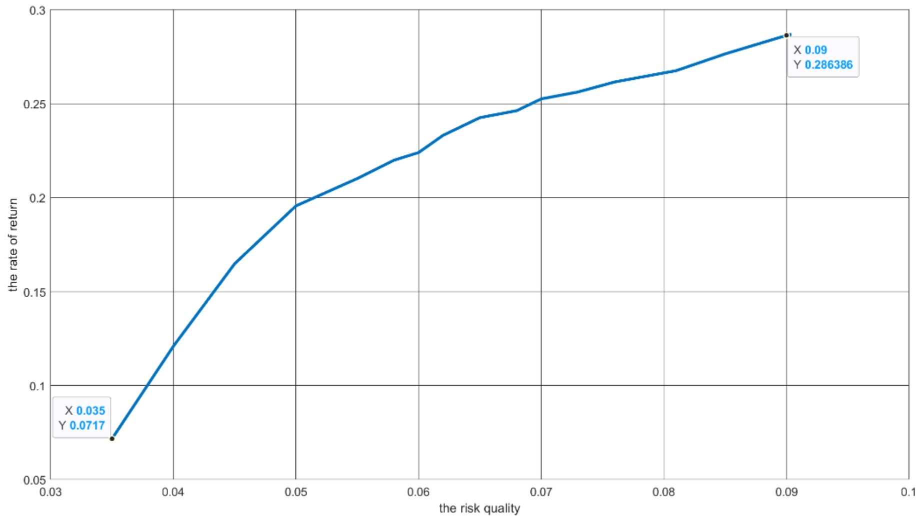

Figure 4. Results of mean-risk quality model operations with realistic constraints

Table 4. Results of the arithmetic data of the mean-risk quality model

RQ	G	The lower semi-standard deviation (10^{-4})
0.04	0.0783	0.6109
0.046778	0.0783	0.6109
0.05	0.0847	0.5243
0.057	0.1812	0.4025
0.06	0.2267	0.8647
0.063	0.2658	1.2667
0.065	0.2904	1.6633
0.0657	0.2989	1.7940
0.07	0.2989	1.7940

Note: Assuming that each stock return is independently and identically distributed, the lower semi-standard deviation of the portfolio $sd_p = \sqrt{(\sum_{i=1}^{n} x_i sd_i)^2}$, where sd_p denotes the standard deviation of the downward bias of the portfolio and sd_i de notes the standard deviation of the downward bias of individual stocks.

5. Discussion

5.1 Comparison and Analysis of Data Results of Risk Quality Model and Risk Index Model

In the preceding section, we explored the capacity of risk quality to estimate downside risk. This was done by comparing the results of data affected by downward-biased standard deviation, and by validating the mean-at-risk quality portfolio selection model. Some limitations, however, exist in using expert-estimated data with this deviation to reflect the risk quality's ability to gauge downside risk. To overcome this, we reference the mean-risk index model established by Huang (2012). We also integrate realistic constraints that characterize the financial market into this model. Finally, we juxtapose this adapted mean-risk index model with the mean-risk quality model to test their effectiveness in estimating downside risk.

An analysis of the tabular data leads to the following conclusions:

- Risk quality and the risk index possess similar capabilities in estimating the downside risk of a portfolio.
- Although risk quality differs from the risk index's estimation of downside risk in terms of significance, it excels in this area as determined by the lower semi-standard deviation.
- Risk quality estimates downside risk in a unique manner, combining uncertainty measures of equity returns with volatility weights. This approach offers a more nuanced expression of the downside risk of equity returns, unlike the risk index.

These findings suggest that investors have the flexibility to choose a risk measurement tool that aligns with their specific risk definitions and needs.

Table 5. Data results of the mean-risk index model

G	The risk index	The lower semi-standard deviation (10^{-4})
0.0532	0.0366	0.1202
0.06	0.0400	0.1001
0.1	0.0440	0.1226
0.12	0.0458	0.1747
0.14	0.0479	0.2572
0.15	0.0489	0.3094
0.17	0.0511	0.4360
0.19	0.0547	0.4861
0.23	0.0707	0.6155
0.25	0.0794	0.7096
0.27	0.0901	0.7744
0.2869	0.1004	0.8447

Table 6. Data results of the mean-risk-quality combination model

G	The risk quality	The lower semi-standard deviation(10^{-4})
0.0532	0.0363	0.1202
0.06	0.0393	0.0983
0.08	0.0411	0.1009
0.1	0.0422	0.1226
0.12	0.0433	0.1744
0.14	0.0444	0.2572
0.16	0.0456	0.3691
0.18	0.0485	0.4479
0.2	0.0514	0.5193
0.25	0.0711	0.7021
0.28	0.0867	0.7895
0.2869	0.0902	0.8239

5.2 Sensitivity Analysis

This section examines the model's sensitivity by studying the effects of variations in financing constraints and transaction cost constraints on the model's results. Model (13), a portfolio optimization model, aims to maximize returns under specific risk capacity constraints. When the constrained risk capacity remains constant, alterations in realistic constraints affect only the returns' magnitude without changing the portfolio's formed ratio. Conversely, model (15) seeks to minimize risk, requiring a certain return rate. Adjustments to each parameter impact the entire model during a sensitivity test. Thus, model (15) is employed in this paper for the sensitivity testing.

We first conduct a sensitivity analysis of the financing constraint for the mean-risk quality model with realistic constraints. By incrementally adjusting the return rate on investment in risk-free assets and the interest rate on borrowed funds, we acquire the model's computational results, summarized in Table 7. From our analysis, we conclude:

- With a constant borrowing rate, the portfolio's risk decreases as the investment rate in risk-free assets rises.
- Keeping the borrowing rate fixed, an escalating borrowing rate leads to increasing portfolio risk.

Integrating a quantitative form of financing constraints into the theoretical model through sensitivity analysis illustrates the model's applicability to real financial markets marked by financing constraints. It also highlights that changes in the return on investment of risk-free assets and the risk-free borrowing rate substantially impact portfolio choices. Investors should consider not only stock returns and risks but also fluctuations in the risk-free lending rate and the return on investment of risk-free assets.

Alternatively, we conduct a sensitivity analysis on the transaction cost constraint and display the findings in Table 8. The data reveals that the portfolio's risk quality rises from 0.0505 to 0.0512 as the unit transaction cost increases from 0.001 to 0.005. Therefore, when other conditions are held constant, the portfolio's risk grows with the continual increase in unit transaction cost. Unlike the borrowing constraint's sensitivity, the transaction cost is not influenced by any conditions and always correlates with portfolio risk.

This analysis serves as a reminder to investors to account for transaction costs in stock trading behavior. Changes in unit transaction costs may influence the portfolio selection results, and it is vital to recognize that even minor unit transaction costs, when trading a large volume of stocks, cannot be overlooked.

Table 7. Results of the sensitivity test data for the financing constraints

r_f^b/r_f^l	The risk quality	r_f^b/r_f^l	The risk quality
0.015/0.017	0.0455722	0.009/0.009	0.0803456
0.01/0.017	0.0460125	0.009/0.013	0.080552
0.008/0.017	0.0460620	0.009/0.017	0.0807556
0.005/0.017	0.0461575	0.009/0.021	0.0810112
		0.009/0.023	0.0810402

Note: The required payoff rate G=0.17 for the left-hand-side arithmetic model; the required payoff rate G=0.27 for the right-hand-side arithmetic model.

Table 8. Results of sensitivity test data for transaction cost constraints

Transaction cost	The risk quality
0.001	0.050528
0.002	0.050710
0.003	0.050763
0.004	0.051090
0.005	0.051244

Note: The required payoff ratio G=0.15 is required for the arithmetic model.

6. Conclusion

This paper introduces a novel risk measurement termed "risk quality," based on uncertainty theory, and explores portfolio optimization under realistic constraints within this framework. We start by employing risk quality to estimate portfolio risk, furnishing specific expressions. Next, a mean-risk quality portfolio optimization model is developed, considering uncertainty theory, and augmented with realistic constraints, such as threshold, financing, and transaction cost constraints. Using the Chinese stock market as a case study, we empirically analyze both the mean-risk quality portfolio optimization model and the model with realistic constraints by employing expert-estimated data rather than historical data. We test the validity of the risk quality models, compare them to the mean-risk index model to gauge their ability in estimating portfolio downside risk, and conduct sensitivity analysis. The key findings are as follows:

- Risk Quality as a Novel Estimation Method. Risk quality is an innovative risk estimation method, integrating uncertainty measures with volatility weighting. Theoretical analysis and numerical tests demonstrate that risk quality excels in estimating the downside risk of equity portfolios.
- Comparison with Risk Index. While risk quality has demonstrated proficiency in constraining the lower semi-standard deviation of stocks compared to the risk index, it doesn't necessarily mean it's superior in all contexts. Investors' choice between the risk quality and risk index methods will depend on their risk preferences.
- Importance of Realistic Constraints. The model's results are significantly influenced by threshold, financing, and transaction cost constraints. Therefore, when utilizing the mean-risk quality portfolio optimization model with these constraints, investors must align the model parameters with the actual financial market conditions to enable more efficient portfolio selection.

The work presented only includes portfolio optimization models with specific constraints. Future research can extend this by exploring how the model performs in financial markets with additional challenges like short selling restrictions, liquidity risk, and background risk. This approach could enhance our understanding of the constraints' endogeneity problem and provide deeper insights into portfolio optimization under uncertainty theory.

Acknowledgement

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Availability of data: Data are available from the authors upon request.

Appendix

Proof 1.

$$\begin{aligned}
&\lim_{r\to-\infty}[(r_f-r)\varphi(r)] \\
&=\lim_{r\to-\infty}[-\frac{\pi(r_f-r)\exp(\pi(e-r)/(\sqrt{3}\delta)}{\sqrt{3}\delta(1+\exp(\pi(e-r)/(\sqrt{3}\delta))^2}] \\
&=\lim_{r\to-\infty}[-\frac{\pi(r_f-r)}{\sqrt{3}\delta(\exp(\pi(r-e)/(\sqrt{3}\delta)+\exp(\pi(e-r)/(\sqrt{3}\delta))^2}] \\
&=\lim_{r\to-\infty}[0+\frac{\pi r}{\sqrt{3}\delta(\exp(\pi(r-e)/(\sqrt{3}\delta)+\exp(\pi(e-r)/(\sqrt{3}\delta))^2}] \\
&=\lim_{r\to-\infty}(\frac{\pi}{\sqrt{3}\delta(\exp(\pi(r-e)/(\sqrt{3}\delta)/r+\exp(\pi(e-r)/(\sqrt{3}\delta)/r)^2}) \\
&=0.
\end{aligned} \tag{17}$$

Proof 2.

First, based on uncertainty theory, the normal uncertainty distribution function is$\phi(r)$, and the specific expression is

$$\phi(r)=\frac{1}{(1+exp(\pi(e-r)/(\sqrt{3}\delta))},t\in\Re. \tag{18}$$

Also, assuming that the function of uncertain variables is$\varphi(r)$, its specific expression is

$$\varphi(r)=-\frac{\pi\, exp(\pi(e-r)/(\sqrt{3}\delta)}{\sqrt{3}\delta(1+exp(\pi(e-r)/(\sqrt{3}\delta))^2},r\in(a,r_f). \tag{19}$$

The risk quality of an equity portfolio is then expressed as

$$\begin{aligned}
RQ&=\sum_{i=1}^{n}x_iRQ_i \\
&=\sum_{i=1}^{n}x_i\int_a^{r_f}(r_f-r)\varphi_i(r)dr \\
&=\sum_{i=1}^{n}x_i\int_a^{r_f}r_f\varphi_i(r)dr-\sum_{i=1}^{n}x_i\int_a^{r_f}r\varphi_i(r)dr \\
&=\sum_{i=1}^{n}x_i[r_f\phi_i(r_f)-a\phi_i(a)]-\sum_{i=1}^{n}x_i\int_a^{r_f}r\varphi_i(r)dr \\
&=\sum_{i=1}^{n}x_i[r_f\phi_i(r_f)-a\phi_i(a)]-\sum_{i=1}^{n}x_i[r_f\phi_i(r_f)-a\phi_i(a)]-\sum_{i=1}^{n}x_i\int_a^{r_f}\phi_i(r)dr \\
&=\sum_{i=1}^{n}x_i(a-r_f)\phi_i(a)+\sum_{i=1}^{n}x_i\int_a^{r_f}\phi_i(r)dr.
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\sum_{i=1}^{n}x_i\int_a^{r_f}\phi_i(r)dr \\
&=\sum_{i=1}^{n}x_i\int_a^{r_f}\frac{1}{(1+\exp(\pi(e_i-r)/(\sqrt{3}\delta_i))}dr \\
&=\sum_{i=1}^{n}x_i(r_f-a)\chi+\sum_{i=1}^{n}x_i\int_\chi^\beta(e_i+\frac{\sqrt{3}\delta_i}{\pi}\ln(\frac{\alpha}{1-\alpha}))d\alpha \\
&=\sum_{i=1}^{n}x_i(r_f-a)\chi+\sum_{i=1}^{n}x_i[(\beta-\chi)e_i+\frac{\sqrt{3}\delta_i}{\pi}(\beta\ln\frac{\beta}{1-\beta}+\ln 1-\beta)-(\chi\ln\frac{\chi}{1-\chi}+\ln 1-\chi)].
\end{aligned} \tag{21}$$

It is known that$\phi(r_f)=\beta,\phi(a)=\chi$, It is possible to get:

$$\phi(r_f) = \frac{1}{(1+exp(\pi(e-r_f)/(\sqrt{3}\delta))} = \beta, \phi(a) = \frac{1}{(1+exp(\pi(e-a)/(\sqrt{3}\delta))} = \chi,$$

then, it can be known that:

$$e_i + \frac{\sqrt{3}\delta_i}{\pi} ln(\frac{\beta}{1-\beta}) = r_f, e_i + \frac{\sqrt{3}\delta_i}{\pi} ln(\frac{\chi}{1-\chi}) = a. \tag{22}$$

Therefore, equation (2) can be transformed into:

$$\begin{aligned}\sum_{i=1}^{n} x_i \int_{a}^{r_f} \phi_i(r)dr &= \sum_{i=1}^{n} x_i (r_f - a)\chi + \sum_{i=1}^{n} x_i \int_{\chi}^{\beta} (e_i + \frac{\sqrt{3}\delta_i}{\pi} \ln(\frac{\alpha}{1-\alpha}))d\alpha \\ &== \sum_{i=1}^{n} x_i (r_f - a)\chi + \sum_{i=1}^{n} \frac{\sqrt{3}\delta_i x_i}{\pi} \ln(\frac{1-\chi}{1-\beta}).\end{aligned} \tag{23}$$

Therefore, the risk quality of a portfolio should be:

$$\begin{aligned} RQ &== \sum_{i=1}^{n} x_i (a - r_f)\phi_i(a) + \sum_{i=1}^{n} x_i \int_{a}^{r_f} \phi_i(r)dr \\ &= \sum_{i=1}^{n} x_i (a - r_f)\phi_i(a) + \sum_{i=1}^{n} \frac{\sqrt{3}\delta_i x_i}{\pi} \ln(\frac{1-\chi}{1-\beta}).\end{aligned} \tag{24}$$

Where, $\chi = \frac{exp[\pi(a-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}{1-exp[\pi(a-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}, \beta = \frac{exp[\pi(r_f-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}{1-exp[\pi(r_f-\sum_{i=1}^{n} x_i e_i)/(\sqrt{3}\sum_{i=1}^{n} x_i \sigma_i)]}$.

End of proof.

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