International Journal of Applied Science; Vol. 8, No. 3; 2025
ISSN 2576-7240 E-ISSN 2576-7259
https://doi.org/10.30560/ijas.v8n3p84

The Promotion of Fibonacci Sequences

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Received: August 31, 2025; Accepted: September 11, 2025; Published: September 12, 2025

Abstract

This paper investigates a generalized class of Fibonacci polynomials by extending their recurrence structure to multivariate and multistage settings. Building on the classical univariate Fibonacci polynomial framework, we introduce new recurrence relations and explore the associated coefficient properties. The study establishes unimodality results for the constructed sequences, derives transformation matrix representations, and develops corresponding generating functions. These results not only enrich the algebraic understanding of Fibonacci-type sequences but also provide potential applications in combinatorial enumeration and related areas of discrete mathematics. Open problems are proposed to guide further research on asymptotic behavior and closed-form characterizations of generalized Fibonacci polynomials.

Keywords: fibonacci polynomials, unimodality, recurrence relations

1. Introduction

The Fibonacci sequence and its polynomial extensions have long been recognized as fundamental objects in number theory, combinatorics, and algebraic analysis. Their recurrence structures and combinatorial interpretations have inspired numerous generalizations, including order-k Fibonacci numbers, Pell-type sequences, and multivariate analogues. Such extensions not only broaden the theoretical framework of recurrence relations but also uncover new algebraic and combinatorial properties with potential applications in areas such as coding theory, cryptography, and algorithmic design.

Traditional Fibonacci polynomials satisfy well-known recurrence relations that have been widely studied for their combinatorial interpretations and generating functions. However, most existing research focuses on univariate extensions, leaving the multivariate and multistage generalizations relatively unexplored. Motivated by this gap, the present work investigates new classes of generalized Fibonacci polynomials obtained by incorporating multiple variables and higher-order recurrence terms.

The primary contributions of this paper are threefold. First, we establish new recurrence structures for generalized Fibonacci polynomials and study the unimodality of the associated coefficient sequences. Second, we develop a matrix formulation that provides a more transparent algebraic perspective on their recursive behavior. Third, we derive generating functions for these polynomials and highlight several unresolved problems regarding their closed-form expressions and asymptotic growth.

By extending the classical framework to a more general setting, this study contributes to the ongoing exploration of Fibonacci-type sequences and provides a foundation for future investigations into their combinatorial and analytic properties.

The well-known Fibonacci polynomials satisfy the following recurrence relation

$$F_{n+2}(x) = F_{n+1}(x) + xF_n(x)$$

Or in the relation

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$$

Some promotions are related to the frequency of intervals, like [3]

$$F_{n+m,m}(x) = F_{n+m-1,m}(x) + xF_{n,m}(x)$$

To define a peculiar Fibonacci function sequence from another perspective [4] by defining k sequences of the generalized order-k $F-P$ numbers as shown: for $n>0, m\geq 0$ and $1\leq i\leq k$

$$u_n^i = 2mu_{n-1}^i + u_{n-2}^i \cdots + u_{n-k}^i$$

with initial conditions for $1-k\leq n\leq 0$:

$$u_n^i = \begin{cases} 1 & \text{if } i=1-n \\ 0 & \text{otherwise} \end{cases}$$

Based on the limitations of univariate interval frequency extension, multivariate y and mul- tistage summation terms are introduced to construct a more general polynomial recurrence framework.

We can further promote this foundation by adding several items and introducing a new vari- able

$$G_{n+m,m}(x,y) = G_{n+m-1,m}(x,y) + \sum_{i=1}^{m-2} G_{n+i,m}(x,y) + G_{n,m}(x,y)$$

Its initial value satisfies

$$G_{0,m}(x,y)=0, G_{i,m}(x,y)=1, i=1,2,\cdots,m-1$$

We define the coefficient values for terms with different degrees

Which is

$$G_{n,m}(x,y) = \sum_{j=0}^{\left\lfloor \frac{n-m+1}{2} \right\rfloor} \sum_{i=0}^{\left\lfloor \frac{n-1}{m} \right\rfloor} S_m(n,i,j) x^i y^j$$

All values beyond the defined range are assigned a value of 0, resulting in the following relational expression. i is associated with the power of x, corresponding to the combinatorial counting dimension of the x-terms in the polynomial; j is associated with the power of y, constrained by the parity of $n-m+1$, reflecting the order limit of multivariable interaction

$$S_m(n+m,k,l) = S_m(n+m-1,k,l) + \sum_{i=2}^{m-1} S_m(n+m-i,k,l-1) + S_m(n,k-1,l)$$

Here are the first few polynomials when $y=x, m=3$.

$$G_{0,3}(x)=0, G_{1,3}(x)=1, G_{2,3}(x)=1, G_{3,3}(x)=x+1, G_{4,3}(x)=3x+1$$

$$G_{5,3}(x)=x^2+5x+1, G_{6,3}(x)=5x^2+7x+1, G_{7,3}(x)=x^3+13x^2+9x+1,$$

$$G_{8,3}(x)=7x^3+25x^2+11x+1, G_{9,3}(x)=x^4+25x^3+41x^2+13x+1$$

The coefficients of x with different powers are defined as follows

$$G_{n,m}(x) = \sum_{j=0}^{\left\lfloor \frac{n-m+1}{2} \right\rfloor} T_m(n,i) x^i$$

Table1. $T_m(n,i)$ write in the form of a list

T3(n, i)	i = 0	i = 1	i = 2	i = 3	i = 4	i = 5	i = 6	i = 7
n = 0	0							
n = 1	1							

n = 2	1							
n = 3	1	1						
n = 4	1	3						
n = 5	1	5	1					
n = 6	1	7	5					
n = 7	1	9	13	1				
n = 8	1	11	25	7				
n = 9	1	13	41	25	1			
n = 10	1	15	61	63	9			
n = 11	1	17	85	129	41	1		
n = 12	1	19	113	231	129	11		
n = 13	1	21	145	377	321	61	1	
n = 14	1	23	181	575	681	231	13	
n = 15	1	25	221	883	1289	681	85	1
n = 16	1	27	265	1209	2241	1683	377	15

2. The Study on the Unimodality of Sequences, Transformation Matrices and Generating Functions

Theorem 1. For any fixed n, $T_m(n,i)$ is unimodal. The value of i corresponding to the maxi- mum value in each row increases as n increases, and if each jump is only a jump of 1.

Proof. Start with the proof of two lemmas. Let n_i denote the threshold index where the in- equality switches, and k_i be an auxiliary index for comparison.

Lemma 2. If the conditions are met

$$T_m(k_i,i) \geq T_m(k_i,i+1), k_i < n_i$$

$$T_m(n_i,i) < T_m(n_i,i+1)$$

For any $k_i \geq n_i$, there exists

$$T_m(k_i,i) < T_m(k_i,i+1)$$

Proof. We use mathematical induction on i. First, define the base case: for $i=3$, verify the condition with initial k_1, n_1 satisfying the lemma 3. Solve it using mathematical induction for i. We can know from the induction conditions that

$$T_m(k_{i-1},i-1) \geq T_m(k_{i-1},i), k_{i-1} < n_{i-1}$$

$$T_m(k_{i-1},i-1) < T_m(k_{i-1},i), k_{i-1} \geq n_{i-1}$$

For i, we inherit the structure by shifting indices, reusing the inequality pattern.

$$T_m(n_{i-1}+1,i) \geq T_m(n_{i-1}+1,i+1)$$

$$\sum_{j=t}^{t+m-2} T_m(j,i-1) < \sum_{j=t}^{t+m-2} T_m(j,i), t > n_{i-1}$$

Based on this, we can conclude that there exists n_i that satisfies the condition

$$T_m(n_i+1,i) \geq T_m(n_{i-1}+1,i+1)$$

If $n_i - 1 \leq n_{i-1} + m - 2$, from the inductive hypothesis, we can draw a conclusion

$$\sum_{j=t-m+2}^{t} T_m(j,i-1) < \sum_{j=t-m+2}^{t} T_m(j,i), t > n_{i-1}, n_n - 1 \le t \le n_{i-1} + m - 2$$

So that

$$T_m(k_i,i) < T_m(k_i,i+1), k_i > n_i \square$$

Using the threshold behavior of n_i and the inequality switching at k_i established in lemma 2, we now derive lemma 3 to further constrain the growth of n_i across indices. This step is critical for proving the index monotonicity of the unimodality. Besides, from the proof of the lemma 2, we know that $n_{i+1} > n_i$. We can also obtain the following proposition.

Lemma 3.

$$n_i + 2 \le n_{i+1}$$

Proof. From the proof of Lemma 2, Lemma 3 is obviously.

By Lemma 3, $n_i + 2 \le n_{i+1}$ ensures that the maximum position i strictly increases with n. Combining Lemma 2 and Lemma 3, we complete the proof of Theorem 1.

Theorem 4. For any integer $k > m$, every line of the form $n = i + k$ is unimodal.

Proof. Rewrite the list and the formula as follows:

$$S_m(n,i) = T_m(n+1,i)$$

The recursive relation expression is

$$S_m(n+1,i+1) = S_m(n,i+1) + \sum_{j=n-m+2}^{n} S_m(j,i)$$

Then, it can be proven through a proof similar to that of Theorem 1

To gain a deeper understanding of this sequence [2], we represent its recurrence algebraically via a matrix H_m. The matrix encodes the coefficients of $G_{n,m}(x,y)$ in its powers H_m^n, where:

$$H_m = \begin{pmatrix} 1 & y & \cdots & y & x \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}$$

Through the form of a matrix, the recursive relationship can be described more specifically.

The following expressions are expressed for high-order iterations.

$$H_m^n = \begin{pmatrix} I_1 & I_2 & \cdots & I_m \end{pmatrix}$$

$$H_m^n = \begin{pmatrix} G_{n+1,m}(x,y) \\ G_{n,m}(x,y) \\ \vdots \\ G_{n-m+2,m}(x,y) \end{pmatrix}, I_m = \begin{pmatrix} xG_{n,m}(x,y) \\ xG_{n-1,m}(x,y) \\ \vdots \\ xG_{n-m+1,m}(x,y) \end{pmatrix}$$

$$I_k = \begin{pmatrix} y\sum_{i=n-m+k+1}^{n} G_{i,m}(x,y) + xG_{n-m+k,m}(x,y) \\ y\sum_{i=n-m+k}^{n-1} G_{i,m}(x,y) + xG_{n-m+k-1,m}(x,y) \\ \vdots \\ y\sum_{i=n-2m+k+2}^{n-m+1} G_{i,m}(x,y) + xG_{n-2m+k+1,m}(x,y) \end{pmatrix}, 2 \le k \le m-1$$

Here, $G_{n,m}(x,y)$ represents a polynomial that satisfies the conditions Equation 1 with the initial condition

$$G_{1,m} = 1, G_{i,m} = 0, i \le 0$$

This matrix representation deepens our understanding by translating combinatorial recurrence into linear algebra, a classic technique in enumerative combinatorics.

Corollary 5. [1] Arbitrarily given $s,t \ge m-1$

$$G_{s+t+1,m}(x,y) = G_{s+1,m}(x,y)G_{t+1,m}(x,y) + J + xG_{s,m}(x,y)G_{t-m+2,m}(x,y)$$

$$J = \sum_{j=0}^{m-3}\left(y\sum_{i=s-m+3+j}^{s} G_{i,m}(x,y) + xG_{s-m+2+j,m}(x,y)\right)G_{t-j,m}(x,y)$$

To analyze oscillatory behavior or growth rates of $G_{n,m}(x,y)$, we extend to complex $y=i$ (where $i^2=-1$). This transforms H_m into a complex matrix:

$$H_m = \begin{pmatrix} 1 & i & \cdots & i & x \end{pmatrix}$$

Define $i^2=-1$, if $y=i$, we will get a matrix

$$H_m = \begin{pmatrix} 1 & i & \cdots & i & x \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix}$$

$$H_m^n = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \end{pmatrix} + iB$$

$$\alpha_1 = \begin{pmatrix} xG_{n+1,m}(x,y) \\ xG_{n,m}(x,y) \\ \vdots \\ xG_{n-m+2,m}(x,y) \end{pmatrix}, \alpha_k = \begin{pmatrix} G_{n-m+k,m}(x,y) \\ G_{n-m-1+k,m}(x,y) \\ \vdots \\ G_{n-2m+1+k,m}(x,y) \end{pmatrix}, 2 \le k \le m$$

$$B=\begin{pmatrix} 0 & \sum_{i=n-m+3}^{n} G_{i,m}(x,y) & \sum_{i=n+3}^{n} G_{i,m}(x,y) & \cdots & G_{n,m}(x,y) & 0 \\ 0 & \sum_{i=n-m+2}^{n-1} G_{i,m}(x,y) & \sum_{i=n-m+3}^{n-1} G_{i,m}(x,y) & \cdots & G_{n-1,m}(x,y) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \sum_{i=n-2m+2}^{n-m+1} G_{i,m}(x,y) & \sum_{i=n-2m+5}^{n-m+1} G_{i,m}(x,y) & \cdots & G_{n-m+1,m}(x,y) & 0 \end{pmatrix}$$

To study the entire sequence $\{G_{i,m}(x,y)\}_{i=0}^{\infty}$ collectively, we define the generating function:

$$G_m(x,y,t)=\sum_{i=0}^{\infty} G_{i,m}(x,y)t^i$$

Subsequently, the real and imaginary parts of the complex number can be discussed separately. Define the generating function

$$G_m(x,y,t)=\sum_{i=0}^{\infty} G_{i,m}(x,y)t^i$$

Theorem 6.

$$G_m(x,y,t)=\frac{t-\frac{y}{t-1}\left[(m-3)t^m-\frac{t^m-t^3}{t-1}\right]}{1-t-y\frac{t^m-t^2}{t-1}-xt^m}$$

Proof. Based on the Equation 1, obviously that

$$G_{n+m,m}(x,y)t^{n+m}=tG_{n+m-1,m}(x,y)t^{n+m-1}+t^{n+m}\sum_{i=1}^{m-2} G_{n+i,m}(x,y)+t^m G_{n,m}(x,y)t^n$$

After summing up both sides and simplifying the fractions, the proof can be completed.

Theorem 7. If function sequence satisfies

$$g_{n+2}(x)=g_{n+1}(x)+xg_n(x), g_1(x)=a, g_2(x)=1$$

$$g_n(x)=\sum_{k=0}^{\left[\frac{n}{2}\right]} s_a(n,k)x^k, n\geq 3$$

$$s_a(n,k)=\sum_{j=1}^{k} B_a(k,j)\binom{n+j-k-1}{k-1}$$

so that $s(n,k)$ satisfies

$$s_a(n,k)=\binom{n-k-1}{k-2}a+\binom{n+j-k-1}{k-1}$$

$$\begin{cases} B_a(k,1) = (1-a)\binom{k}{1} + a \\ B_a(k,j) = (-1)^j (a-1)\binom{k}{j} \end{cases}$$

Table 2. **Example 8.** When $a = 2$,

B2(k, j)	j = 1	j = 2	j = 3	j = 4	j = 5	j = 6	j = 7	j = 8
k = 2	0	1						
k = 3	-1	3	-1					
k = 4	-2	6	-4	1				
k = 5	-3	10	-10	5	-1			
k = 6	-4	15	-20	15	-6	1		
k = 7	-5	21	-35	35	-21	7	-1	
k = 8	-6	28	-56	70	-56	28	-8	1

It is triangle T read by rows derived from the signed Pascal triangle and satisfying $T = T^{-1}$.

Proof.

$$g_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \sum_{j=1}^{k} B_a(k,j) \binom{n+j-k-1}{k-1} x^k$$

The problems that we have not solved is that:

Problem 9. How to derive the general term formula of

$$G_{n+m,m}(x,y) = G_{n+m-1,m}(x,y) + \sum_{i=1}^{m-2} G_{n+i,m}(x,y) + G_{n,m}(x,y)$$

Problem 10. For a very large m, are there any other statistical properties of generalized Fibonacci-type polynomials?

5. Conclusion and Discussion

In this work, we have extended the classical Fibonacci polynomial framework by introducing multivariate and multistage recurrence relations. The analysis established unimodality properties of the resulting coefficient sequences, provided a linear-algebraic perspective through matrix representations, and constructed generating functions to capture structural characteristics. These findings reflect the core objectives outlined in the introduction: to broaden the scope of Fibonacci polynomials, reveal new algebraic properties, and lay a foundation for further study of generalized recurrence systems.

The results presented here also align with the broader motivation of enriching the theory of recurrence relations and their combinatorial interpretations. While our focus has been primarily theoretical, potential applications in combinatorial enumeration, discrete probability, and algorithmic design suggest promising directions for future exploration. The open problems posed—such as the derivation of closed-form expressions and the statistical analysis of large-scale polynomial coefficients—represent natural continuations of this work and may uncover deeper structural insights.

Overall, this study contributes incremental but meaningful progress to the field of generalized Fibonacci-type polynomials. By providing both new results and open research avenues, it bridges the classical theory of recurrence relations with more modern, multivariate approaches. We hope that the framework developed here will inspire further investigations and applications across mathematics and its related disciplines.

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