International Journal of Applied Science; Vol. 7, No. 2; 2024
ISSN 2576-7240 E-ISSN 2576-7259
https://doi.org/10.30560/ijas.v7n2p93

Uniform Decay for the Viscoelastic Kirchhoff-Type Equations with Memory and Delay Terms

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Received: December 14, 2024; Accepted: December 30, 2024; Published: December 31, 2024

Abstract

In this paper, we investigate the initial boundary value problem of the viscoelastic Kirchhoff-type equations with memory and delay terms. We investigate the initial boundary value problem of viscoelastic Kirchhoff-type equations with memory and delay terms. The analysis assumes relatively weak conditions for the relaxation function g. This paper introduces a new framework for analyzing decay properties under weaker conditions, improving upon previous results in the literature. Finally, we prove that the system energy exhibits exponential and polynomial decay rates, which depend on the behavior of the relaxation function g. This provides new insights into the energy decay properties of Kirchhoff-type viscoelastic wave equations under weaker conditions on the relaxation function, and provides a valuable technical framework for exploring the decay properties of more complex partial differential equations.

Keywords: viscoelastic kirchhoff-type equations, uniform decay, viscoelastic damping, time delay, relaxation function

1. Introduction to the Problem and Background

The present paper provides a detailed analysis of viscoelastic Kirchhoff-type equations, considering memory effects and time-varying delays:

$$\begin{cases} u_{tt}-M(\|\nabla u\|^2)\Delta u+\int_0^t g\left(t-s\right)\Delta u\left(s\right)\,ds+\mu_1 u_t(x,t)+\mu_2 u_t(x,t-\tau(t))=0, \\ (x,t)\in\Omega\times(0,+\infty), \\ u(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,+\infty), \\ u\left(x,0\right)=u_0\left(x\right), \quad u_t\left(x,0\right)=u_1\left(x\right), \quad x\in\Omega, \\ u_t(x,t)=f_0(x,t), \quad x\in\Omega, \quad t\in\left[-\tau(0),0\right). \end{cases} \tag{1.1}$$

where $M(\|\nabla u\|^2)$ represents the nonlinear elastic response of the material, depending on the inherent characteristics of the material. $M(s) = 1+s^{\gamma}$, $\gamma, s \geq 0$, and $\Omega \in \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. $g(t)$ is a monotonically decreasing positive function physically signifies that the memory effect in viscoelastic materials weakens over time. The term $\int_0^t g(t-s)\Delta u(s)\,ds$ represents the viscoelastic damping term, reflecting the material's memory effect. $\mu_1 u_t(x, t)$ represents the frictional damping in the model, which used to capture the mechanism of energy dissipation. $\mu_2 u_t(x, t-\tau(t))$ represents the delayed damping term, which describes the system's response due to delayed feedback. $\tau(t)$ denotes the time delay, and μ_1, μ_2 are constants. The specific constraints will be provided with detailed analysis and explanation in the following sections. The boundary condition $u(x, t) = 0$ describes a system with fixed boundaries. For example, in the context of vibration analysis, a string or beam with fixed endpoints is typically modeled with the boundary condition $u(x, t)=0$ at the endpoints. This condition ensures that the displacement of the string or beam is zero at the fixed points, which realistically reflects how materials are often constrained in engineering applications, such as in musical instruments or structural components. u_0, u_1 and f_0 are the given initial conditions.

Viscoelastic systems have long been studied for their ability to model complex physical phenomena involving memory and delay effects. While prior studies imposed strong conditions on the relaxation function, this paper seeks to relax these assumptions, providing broader applicability. There are various specific forms of the study of Kirchhoff-type viscoelastic equations. For a comprehensive overview, please refer to [1–5, 7, 8, 10–12]. In the study of viscoelastic equations, the occurrence of time delay is a recurring and critical phenomenon. The control problem of partial differential equations with time delay effects plays an important role in real-world applications, as time delay phenomena are present in many physical, engineering, and biological systems. Many researchers add time-delay frictional damping terms to Kirchhoff-type viscoelastic equations in order to study the problems in greater detail. Relevant references can be found in [4, 8, 10, 11].

In the study of problems without viscoelastic damping terms, early research has established numerous results on global existence, decay behavior, and blow-up phenomena. For example, equations of the following form:

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + h(u_t) = f(u).$$

When $M = 1$ and with the presence of the viscoelastic damping term, X.G. Yang [6] studied the stability of a class of nonlinear viscoelastic systems with time delay terms

$$u_{tt} - \Delta u + \int_0^t g(t-s)\,\Delta u(s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)) = h(x)$$

Yang [6] established exponential decay under stringent conditions. By contrast, our work aims to achieve similar results with weaker assumptions.

In the study of the uniform decay properties of Kirchhoff-type viscoelastic equations, imposing stronger restrictions on the relaxation function leads to more stringent criteria for selecting viscoelastic materials in the system. Relaxing these restrictions appropriately holds significant importance for advancing the understanding of viscoelastic systems. K. Daewook [4] studied the uniform decay properties of Kirchhoff-type energy

$$\begin{aligned} &u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t h(t-s)\,div[a(x)\nabla u(\tau)]\,d\tau + |u|^{\gamma} u \\ &+\mu_1 u_t(x,t) + \mu_2 u_t(x, t-s(t)) = 0. \end{aligned}$$

The system model describes the axial motion of viscoelastic materials, where under the restrictions on the relaxation function such that $-\xi_1 \le h'(t) \le -\xi_2 h(t)$ and $0 \le h''(t) \le \xi_3 h(t)$, it is shown that the system energy exhibits an exponential decay form.

In a class of weakly damped viscoelastic Kirchhoff-type wave equations, J. B. Zuo [10] and others studied a model with Balakrishnˆan-Taylor damping and variable exponent nonlinear time delay terms

$$\begin{aligned} &u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \alpha(t)\int_0^t g(t-s)\,\Delta u(s)\,ds + \mu_1 |u_t(t-\tau)| \\ &+\mu_2 |u_t(t-\tau(t))|^{p(x)-2} u_t(t-\tau(t)) = 0. \end{aligned}$$

The restriction on the relaxation function is $g'(t) \le -\xi(t)g(t)$. Under this condition, it is shown that the system energy exhibits an exponential decay form, rather than a polynomial decay result.

When there is no time-delay friction damping term, E. Piskin [12] and others studied the system.

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\,\Delta u(s)\,ds = u \ln|u|,$$

the form of the relaxation function restricted here is $g'(t) \le -\xi g(t)$, which leads to an exponential decay result for the system energy.

We study the uniform decay of a class of viscoelastic Kirchhoff-type equations with memory and delay terms, controlling the relaxation function $g(t)$ under weaker conditions, and investigate its corresponding optimal decay rate. Yang [6] and Zuo [10] explored stability under different damping conditions, which this work generalizes. The highlight of this paper is the improvement of earlier decay results, introducing key technical tools, and optimizing the cumbersome calculations in the process of solving partial differential equations. Our approach allows for faster derivation of decay results, which was not achievable in earlier works. For the theoretical guidance related to this study, interested readers can refer to references [21, 22]. The constants C_i, $C(i) > 0$ appearing in the paper represent some positive constants, with their positions indicating different numerical values.

2. Uniform Decay Result

This section establishes uniform decay results by constructing energy functionals, applying integral inequalities, and leveraging Lyapunov techniques. This work utilizes the standard Lebesgue and Sobolev spaces along with their associated inner products and norms. We impose the following conditions on the relaxation function *g(t)*:

(H1) There exists a positive non-increasing differentiable function *ξ(t)*, such that the continuously differentiable function *g(t)* satisfy the following conditions:

$$1-\int_0^{\infty} g(s)\,ds = l > 0,\quad g'(t) \le -\xi(t)\,g^p(t),\quad t\in[0,+\infty),$$

here, $1 \le p < 2$, which indicates that $0 < l < 1$ is a constant.

Remark 2.1. This condition ensures that the relaxation function decays at a controlled rate.

We provide several examples of relaxation functions *g(t)*:

Example 1: When $p = 1$, the relaxation function is given by $g(t) = le^{-t}$, subject to $\xi(t) = 1$.

Example 2: When $p = 1$, the relaxation function is given by $g(t) = \frac{(1-l)l}{(t+1)^{l+1}}$, subject to $\xi(t) = \frac{1+l}{t+l}$.

Example 3: When $1 < p < 2$, the relaxation function is given by $g(t) = C_{10}e^{-kt^{p-1}}$

for $k > 0$ and $C_{10} = \frac{(1-l)(P-1)k^{\frac{1}{P-1}}}{\Gamma\left(\frac{1}{P-1}\right)}$, , subject to $\xi(t) = \frac{C_{10}k(p-1)}{C_{10}^{P}}t^{p-2}$.

(H2) Suppose that $\sqrt{(1-\tau')}\mu_1 > \mu_2 > 0$ and $\tau(t) \in C^1[0,+\infty]$ satisfies the following relationship:

$$0 < \tau'(t) < 1,\ \forall t \ge 0.$$

These conditions allow us to construct appropriate energy functions based on the structure of system (1.1). The energy of system (1.1) is defined as:

$$E(t) := \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{\xi}{2}\int_\Omega\int_0^1 u_t^2(x,t-\tau(t)\eta)\,d\eta\,dx, \tag{2.1}$$

where

$$\frac{\tau\mu_2^2}{(1-\tau')\mu_1} < \xi < \tau\mu_1,\quad (g\circ v)(t) = \int_0^t g(t-s)\|v(t)-v(s)\|_2^2\,ds.$$

The coefficient γ controls how a material's stiffness changes with deformation. When $\gamma = 1$, the material behaves elastically (linear response). For $\gamma > 1$, the material hardens, becoming stiffer as it deforms, while for $\gamma < 1$, it softens, with stiffness decreasing as deformation increases. This parameter significantly impacts the material's mechanical response and the design of structures. ξ is a positive constant we selected to ensure that the constructed energy function exhibits a decreasing behavior.

Remark 2.2. *If a function f is equivalent to a function g, it can be denoted as f ~ g, which means there exist positive constants c_1 and $c_2 > 0$ such that: $c_1 f \le g \le c_2 f$. It can be seen from (H1) that E(t) > 0 and*

$$E(t) \sim \|u_t\|_2^2 + \|\nabla u\|_2^2 + (g\circ\nabla u)(t) + \int_\Omega\int_0^1 u_t^2(x,t-\tau(t)\eta)\,d\eta\,dx.$$

Lemma 2.1. *Given condition (H1), if the initial conditions satisfy $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then the system (1.1) admits a unique weak solution u, implying that:*

$$u \in L^\infty(0,+\infty; H_0^1(\Omega)),\ u_t \in L^\infty(0,+\infty; L^2(\Omega)).$$

Remark 2.3. *For the proof of existence, interested readers can refer to [13–15]. Lemma 2.1 ensures the existence of a unique weak solution, providing a foundation for the energy decay analysis.*

Theorem 2.4. *If $(u_0, u_1) \in H_2^1(\Omega)\times L^2(\Omega)$ and the conditions (H1)−(H2) are satisfied, hen the energy E(t) of the solution to system (1.1) satisfies*

• *If $p = 1$, then there exists a constant $k_0 > 0$, such that*

$$E(t) \;\leq\; CE(0)e^{-k_0\int_0^t \xi(s)\,ds}, \quad t \geq 0. \tag{2.2}$$

• *If $1 < p < 2$, then there exist two constants $k_1, k_2 > 0$, such that*

$$E(t) \;\leq\; CE(0)\frac{1}{\left(1+\int_0^t \xi(s)ds\right)^{\frac{1}{p-1}}}, \quad t \geq 0. \tag{2.3}$$

Remark 2.5. *Exponential decay is only effective under the condition of $p = 1$. Theorem 2.4 provides the specific form of system energy decay, which provides correct theoretical guidance for subsequent research. Theorem 2.4 is proposed based on the theoretical guidance of previous works, and interested readers can refer to [20].*

3. Construction and Proof of the Auxiliary Function

Lemma 3.1. *In order to better study the decay behavior of the system, we introduce new variables to replace the delay term,*

$$z(x, \eta, t) = u_t(x, t - \eta\tau(t)),$$

and $z(x, \eta, t)$ satisfies the following

$$\tau z_t(x, \eta, t) + (1 - \eta\tau')z_\eta(x, \eta, t) = 0.$$

Proof. $z_t(x, \eta, t)$ is the partial derivative of $z(x, \eta, t)$ with respect to t. By applying the chain rule of differentiation, we have

$$z_t(x,\eta,t) = \frac{\partial}{\partial t}u_t(x, t-\eta\tau(t)) = u_{tt}(x, t-\eta\tau(t))\cdot\left(1-\eta\tau'(t)\right).$$

We take the partial derivative of $z(x, \eta, t)$ with respect to η,

$$z_\eta(x,\eta,t) = \frac{\partial}{\partial \eta}u_t(x, t-\eta\tau(t)) = -u_{tt}(x, t-\eta\tau(t))\cdot\tau(t).$$

We can complete the proof by combining the above expressions.

Lemma 3.2. If u is a solution to equation (1.1), then we have

$$\begin{aligned} E'(t) \;\leq\; & -C_1\left(\int_\Omega u_t^2(t)\,dx + \int_0^1\int_\Omega z^2(x,\eta,t)\,dx\,d\eta + \int_\Omega z^2(x,1,t)\,dx\right) \\ & +\frac{1}{2}\left(g'\circ\nabla u\right)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2 dx. \end{aligned} \tag{3.1}$$

Proof. We differentiate the energy function $E(t)$, and using equation (1.1) along with the integration by parts method, we obtain

$$\begin{aligned} E'(t) &= \int_\Omega u_t u_{tt}\,dx - \frac{1}{2}g(t)\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\left(1-\int_0^t g(s)\,ds\right)\int_\Omega 2\nabla u\cdot\nabla u_t\,dx \\ &+ \|\nabla u\|^{2\gamma}\int_\Omega \nabla u \nabla u_t\,dx + \xi\int_\Omega\int_0^1 z(x,\eta,t)z_t(x,\eta,t)\,d\eta\,dx \\ &+\frac{1}{2}(g'\circ\nabla u)(t) + \frac{1}{2}\int_\Omega\int_0^t g(t-s)\,2(\nabla u(t)-\nabla u(s))\cdot\nabla u_t\,ds\,dx \\ &= \overbrace{\int_\Omega u_t u_{tt}\,dx}^{A_1} - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \left(1-\int_0^t g(s)\,ds\right)\int_\Omega \Delta u\cdot u_t\,dx \\ &+\frac{1}{2}(g'\circ\nabla u)(t) - \int_\Omega u_t\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))\,ds\,dx \\ &+ \|\nabla u\|^{2\gamma}\int_\Omega \nabla u\nabla u_t\,dx + \xi\overbrace{\int_\Omega\int_0^1 z(x,\eta,t)z_t(x,\eta,t)\,d\eta\,dx}^{A_2}. \end{aligned} \tag{3.2}$$

Using equation (1.1) and analyzing the A_1 term, we have

$$\begin{aligned} A_1 &= \int_\Omega u_t(t)M(\|\nabla u\|^2)\Delta u(x,t)\,dx - \int_\Omega u_t\int_0^t g(t-s)\Delta u(x,s)\,ds\,dx \\ &-\mu_1\int_\Omega u_t^2(t)\,dx - \mu_2\int_\Omega u_t(t)z(x,1,t)\,dx \\ &= \int_\Omega u_t(t)\Delta u + u_t(t)\|\nabla u\|^{2\gamma}\Delta u\,dx - \int_\Omega u_t\int_0^t g(t-s)\Delta u(x,s)\,ds\,dx \\ &-\mu_1\int_\Omega u_t^2(t)\,dx - \mu_2\int_\Omega u_t(t)z(x,1,t)\,dx. \end{aligned} \tag{3.3}$$

By applying Lemma 3.2 to analyze the last term A_2, we obtain

$$\begin{aligned} A_2 &= -\int_0^1\int_\Omega \frac{1-\eta\tau'}{\tau}z(x,\eta,t)z_\eta(x,\eta,t)\,dx\,d\eta \\ &= -\frac{1}{2}\int_\Omega\int_0^1 \frac{1-\eta\tau'}{\tau}\,d(z^2(t))\,dx \\ &= -\frac{1}{2}\int_\Omega [\frac{1-\eta\tau'}{\tau}z^2(t)]|_0^1\,dx + \frac{1}{2}\int_\Omega\int_0^1\left(\frac{\partial}{\partial\eta}\frac{1-\eta\tau'}{\tau}\right)z^2(t)\,d\eta\,dx \\ &= -\frac{1-\tau'}{2\tau}\int_\Omega z^2(x,1,t)\,dx + \frac{1}{2\tau}\int_\Omega u^2(t)\,dx - \frac{\tau'}{2\tau}\int_0^1\int_\Omega z^2(x,\eta,t)\,dx\,d\eta. \end{aligned} \tag{3.4}$$

By combining equations (3.2)-(3.4), we arrive at

$$\begin{aligned} E'(t) &= -\frac{1}{2}g(t)\|\nabla u\|_2^2 - (\mu_1 - \frac{\xi}{2\tau})\int_\Omega u_t^2(t)\,dx - \frac{\xi(1-\tau')}{2\tau}\int_\Omega z^2(x,1,t)\,dx \\ &-\frac{\xi\tau'}{2\tau}\int_0^1\int_\Omega z^2(x,\eta,t)\,dx\,d\eta - \mu_2\int_\Omega u_t(t)z(x,1,t)\,dx + \frac{1}{2}(g'\circ\nabla u)(t) \end{aligned} \tag{3.5}$$

The proof of Lemma 3.2 can be completed based on *S*. Nicaise [19, Proposition 2.4].

Remark 3.1. *From this lemma, we can observe that the energy* $E(t)$ *of the system follows a decreasing trend. However, the rate of decrease is influenced by the relationship between the parameters* μ_1 *of the friction damping term and* μ_2 *of the delay term in the system.*

In order to control the potential energy term $\|\nabla u(t)\|$ in the system energy $E(t)$, we introduce the following auxiliary function $K_1(t)$:

$$K_1(t) := \int_\Omega u u_t\,dx \tag{3.6}$$

Lemma 3.3. *If (H1) − (H2) holds, then we have*

$$K_1'(t) \leq -\frac{l}{4}\|\nabla u(t)\|_2^2 + C_2\left(\|u_t\|_2^2 + \|u_t(x,t-\tau(t))\|_2^2\right) + \frac{1}{2l}\int_\Omega \left(\int_0^t g(t-s)\,|\nabla u(s) - \nabla u(t)|\,ds\right)^2 dx, \quad t \geq 0 \tag{3.7}$$

where $C_2 > 0$ is a positive constant.

Proof. Direct computations, along the solution of (1.1), and applying integration by parts, we get

$$\begin{aligned} K_1'(t) &= \int_\Omega \left(uu_{tt} + u_t^2\right) dx \\ &= \int_\Omega u_t^2\,dx + \int_\Omega uM(\|\nabla u\|^2)\Delta u\,dx - \int_\Omega u\int_0^t g(t-s)\,\Delta u(s)\,ds\,dx \\ &\quad - \int_\Omega u\mu_1 u_t\,dx - \int_\Omega u\mu_2 u_t(x,t-\tau(t))\,dx \\ &= \int_\Omega u_t^2\,dx - M(\|\nabla u\|^2)\int_\Omega |\nabla u|^2\,dx + \overbrace{\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\,\nabla u(s)\,ds\,dx}^{A_3} \\ &\quad \overbrace{-\int_\Omega u\mu_1 u_t\,dx - \int_\Omega u\mu_2 u_t(x,t-\tau(t))\,dx}^{A_4}. \end{aligned} \tag{3.8}$$

Applying Young's inequality to estimate the term A_3 on the right-hand side of equation (3.8), we obtain

$$\begin{aligned} A_3 &= \int_0^t g(s)\,ds\int_\Omega |\nabla u(t)|^2\,dx + \int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\left(\nabla u(s) - \nabla u(t)\right)ds\,dx \\ &\leq \int_0^t g(s)\,ds\int_\Omega |\nabla u(t)|^2\,dx + \frac{l}{2}\int_\Omega |\nabla u|^2\,dx \\ &\quad + \frac{1}{2l}\int_\Omega \left(\int_0^t g(t-s)\,|\nabla u(s) - \nabla u(t)|\,ds\right)^2 dx \\ &\leq \left(1 - \frac{l}{2}\right)\int_\Omega |\nabla u(t)|^2\,dx + \frac{1}{2l}\overbrace{\int_\Omega \left(\int_0^t g(t-s)\,|\nabla u(s) - \nabla u(t)|\,ds\right)^2 dx}^{A_5}. \end{aligned} \tag{3.9}$$

Where the condition $\int_0^t g(s)\,ds \leq \int_0^\infty g(s)\,ds = 1 - l$ has been applied.

By applying Poincaré's inequality and Young's inequality to the term A_4, we obtain, for any $\delta_1 > 0$, the following

$$\begin{aligned} |A_4| &\leq \mu_1\left(\int_\Omega u_t^2\,dx\right)^{\frac{1}{2}}\left(\int_\Omega u^2\,dx\right)^{\frac{1}{2}} + \mu_2\left(\int_\Omega z^2(x,1,t)\,dx\right)^{\frac{1}{2}}\left(\int_\Omega u^2\,dx\right)^{\frac{1}{2}} \\ &\leq \mu_1\left(\int_\Omega u_t^2\,dx\right)^{\frac{1}{2}}\left(C_p\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}} + \mu_2\left(\int_\Omega z^2(x,1,t)\,dx\right)^{\frac{1}{2}} C_p^{\frac{1}{2}}\|\nabla u\|_2^2 \\ &\leq \left(C_p\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}}\left[\left(\mu_1\int_\Omega u_t^2\,dx\right)^{\frac{1}{2}} + \left(\mu_2\int_\Omega z^2(x,1,t)\,dx\right)^{\frac{1}{2}}\right] \\ &\leq \delta_1 C_p\int_\Omega |\nabla u|^2\,dx + \frac{1}{2\delta_1}\left(\mu_1\int_\Omega u_t^2\,dx + \mu_2\int_\Omega z^2(x,1,t)\,dx\right). \end{aligned} \tag{3.10}$$

By combining (3.8)-(3.10), we arrive at

$$K_1'(t) \leq -\left(\frac{l}{2} + \|\nabla u\|^{2\gamma} - \delta_1 C_p\right)\|\nabla u\|_2^2 + \left(1 + \frac{\mu_1}{2\delta_1}\right)\|u_t\|_2^2 + \frac{\mu_2}{2\delta_1}\|z(x,1,t)\|_2^2 + \frac{1}{2l}A_5. \tag{3.11}$$

By utilizing the boundedness of $\|\nabla u\|_2^{2\gamma}$, we know that

$$\|\nabla u\|_2^{2\gamma} \leq \left(\frac{2}{l}E(t)\right)^{\gamma} \leq \left(\frac{2}{l}E(0)\right)^{\gamma}$$

therefore, we obtain

$$K_1'(t) \leq -\left(\frac{l}{2} - \delta_1 C_p\right)\|\nabla u\|_2^2 + \left(1 + \frac{\mu_1}{2\delta_1}\right)\|u_t\|_2^2 + \frac{\mu_2}{2\delta_1}\|z(x,1,t)\|_2^2 + \frac{1}{2l}A_5 \tag{3.12}$$

By choosing $\delta_1 = \frac{l}{4C_p}$ in the above inequality (3.12), we complete the proof.

Lemma 3.4. *The Lyapunov auxiliary function L(t) is constructed for*

$$L(t) = NE(t) + K_1(t).$$

By selecting appropriate constants $N > 0$ and $C_3, C_4 > 0$ such that $C_3E(t) \leq L(t) \leq C_4E(t)$, there exists a positive constant $\epsilon_0 > 0$, we obtain

$$L'(t) \leq -\epsilon_0\left(\|\nabla u(t)\|_2^2 + \|u_t\|_2^2 + \int_0^1\int_\Omega u_t^2(x, t-\eta\tau(t))\,dx\,d\eta\right) + \frac{1}{2l}A_5. \tag{3.13}$$

Proof. From the definition of $K_1(t)$, it is easy to observe that

$$|K_1(t)| \leq CE(t).$$

Where C is a positive constant, and the detailed proof can be found in [9, Lemma 3.2]. Therefore, we can choose a sufficiently large $N > 1$ such that for some constants $C_3, C_4 > 0$ the inequality $C_3E(t) \leq L(t) \leq C_4E(t)$ holds. Furthermore, by combining Lemmas 3.1, 3.2, and 3.3, we can conclude that:

$$\begin{aligned} L'(t) \leq& -\frac{l}{4}\|\nabla u(t)\|_2^2 - (C_1N - C_2)\left(\|u_t\|_2^2 + \|u_t(x, t-\tau(t))\|_2^2\right) \\ &+ \frac{1}{2}N(g' \circ \nabla u)(t) + \frac{1}{2l}A_5. \end{aligned}$$

Therefore, by choosing $N = \frac{2C_2}{C_1}$, The proof of the lemma 3.4 has been completed. Building on this framework, we proceed to establish decay estimation in the next section.

4. Proof of theorem 2.4

The work we have done so far serves as a preparation for proving the uniform decay result of Theorem 2.4. In this section, we will provide the detailed proof process.

Proof. First, let

$$C_\beta = \int_0^t \frac{g^2(s)}{\beta g(s) - g'(s)}\,ds, \quad h(t) = \beta g(t-s) - g'(t-s),$$

where $\beta > 0$ is an undetermined constant.

By using Hölder inequality to estimate the term A_5, we have

$$\begin{aligned} A_5 &= \int_\Omega\left(\int_0^t \frac{g(t-s)}{\sqrt{\beta g(t-s) - g'(t-s)}}\sqrt{\beta g(t-s) - g'(t-s)}\,|\nabla u(s) - \nabla u(t)|\,ds\right)^2 dx \\ &\leq \left(\int_0^t \frac{g^2(s)}{\beta g(s) - g'(s)}\,ds\right)\int_\Omega\int_0^t[\beta g(t-s) - g'(t-s)]\,|\nabla u(s) - \nabla u(t)|^2\,ds\,dx \\ &\leq C_\beta\int_0^t h(t-s)\,\|\nabla u(s) - \nabla u(t)\|_2^2\,ds \\ &\leq C_\beta(h \circ \nabla u)(t). \end{aligned} \tag{4.1}$$

By Lemma 3.4 and equation (4.1), we obtain

$$\begin{aligned} L'(t) \leq& -\epsilon_0\left(\|\nabla u(t)\|_2^2 + \|u_t\|_2^2 + \int_0^1\int_\Omega u_t^2(x, t-\eta\tau(t))\,dx\,d\eta\right) \\ &+ \frac{1}{2l}C_\beta(h \circ \nabla u)(t). \end{aligned} \tag{4.2}$$

The next step is to continue handling and controlling the term A_5. To do this, we need to introduce a key auxiliary function $K_3(t)$, which was first proposed by Jin-Liang-Xiao in [16]. This auxiliary function can help us obtain the integrability of system energy $E(t)$ on [0,+∞).

$$K_3(t) = \beta \int_0^t \Phi(t-s) \|\nabla u(s)\|_2^2 \, ds + E(t).$$

Here, $\Phi(t) = \int_t^\infty g(s)\,ds$, and it can be observed that $\Phi(0) = \int_0^\infty g(s)\,ds = 1 - l < 1$. According to the references [16–18], it has the following properties

$$\begin{aligned} \frac{d}{dt}K_3(t) &= \beta\Phi(0)\|\nabla u(t)\|_2^2 - \beta\int_0^t g(t-s)\|\nabla u(s)\|_2^2\,ds + E'(t) \\ &\leq 2\beta\|\nabla u(t)\|_2^2 - \frac{1}{2}\int_0^t (\beta g(t-s) - g'(t-s))\|\nabla u(t) - \nabla u(s)\|_2^2\,ds \\ &\leq 2\beta\|\nabla u(t)\|_2^2 - \frac{1}{2}(h \circ \nabla u)(t). \end{aligned} \tag{4.3}$$

Therefore, we set

$$\widetilde{L}(t) = L(t) + \frac{C_\beta}{l}K_3(t).$$

By combining equations (4.2) and (4.3), we obtain

$$\widetilde{L}'(t) \leq -\left(\epsilon_0 - \frac{2\beta C_\beta}{l}\right)\left(||\nabla u(t)||_2^2 + ||u_t||_2^2 + \int_0^1\int_\Omega u_t^2(x, t-\eta\tau(t))\,dx\,d\eta\right).$$

As known from [17, page 1525], we have

$$\beta C_\beta \to 0,\ \beta \to 0.$$

Therefore, by choosing an appropriate value for $\beta = \frac{l\epsilon_0}{4C_\beta}$, we obtain

$$\widetilde{L}'(t) \leq -\frac{1}{2}\epsilon_0\left(||\nabla u(t)||_2^2 + ||u_t||_2^2 + \int_0^1\int_\Omega u_t^2(x, t-\eta\tau(t))\,dx\,d\eta\right). \tag{4.4}$$

Moreover, by definition, we have $\tilde{L}(t) \geq 0$ and $\tilde{L}(t_0) \leq CE(0)$. Integrating equation (4.4) over $[t_0, T]$, we obtain

$$\begin{aligned} &\int_{t_0}^T \left(||\nabla u(t)||_2^2 + ||u_t||_2^2 + \int_0^1\int_\Omega u_t^2(x, t-\eta\tau(t))\,dx\,d\eta\right)dt \\ &\leq -C\int_{t_0}^T \widetilde{L}'(t)dt \\ &\leq C\widetilde{L}(t_0) \leq CE(0). \end{aligned}$$

The above equation is uniform with respect to T, therefore, we obtain

$$\int_0^{+\infty}\left(||\nabla u(t)||_2^2 + ||u_t||_2^2 + \int_0^1\int_\Omega u_t^2(x, t-\eta\tau(t))\,dx\,d\eta\right)dt \leq CE(0).$$

Furthermore, by the properties of double integrals, we can obtain

$$\int_0^{+\infty}\int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^2 ds dt \leq C\int_0^{+\infty}\|\nabla u\|_2^2 dt \leq CE(0).$$

We note Remark 2.2, which implies that the energy $E(t)$ of the solution to system (1.1) is integrable, that is

$$\int_0^{+\infty} E(t)dt \leq CE(0). \tag{4.5}$$

Since $E'(t) \leq 0$, by the fundamental properties of integration, we can obtain

$$E(t) \leq CE(0)(t+1)^{-1},\ t \geq 0. \tag{4.6}$$

Furthermore, by Hölder inequality, we know that

$$\begin{aligned}A_5 &\leq \int_0^t g(s)ds \int_\Omega \int_0^t g(t-s)\left|\nabla u(s)-\nabla u(t)\right|^2 ds\,dx \\ &\leq \int_0^t g(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|_2^2 ds \\ &\leq (g\circ\nabla u)(t).\end{aligned}$$

Thus, using the above expression along with equation (3.13) and Remark 2.2 we obtain that

$$\begin{aligned}L'(t) \leq & -\epsilon_0\left(\|u_t\|_2^2+\|\nabla u\|_2^2+\int_0^1\int_\Omega u_t^2(x,t-\eta\tau(t))\,dx\,d\eta+(g\circ\nabla u)(t)\right) \\ & +\left(\frac{l}{2}+\epsilon_0\right)(g\circ\nabla u)(t), \quad t\geq t_0 \end{aligned} \tag{4.7}$$

holds.

Additionally, by Lemma 3.4, we know that

$$L(t)\sim E(t)\sim \|u_t\|_2^2+\|\nabla u\|_2^2+(g\circ\nabla u)(t)+\int_0^1\int_\Omega u_t^2(x,t-\eta\tau(t))\,dx\,d\eta.$$

Therefore, from (4.7), we have, for some $\tilde{\epsilon}_0$, $\tilde{C}_5>0$,

$$L'(t)\leq -\widetilde{\epsilon}_0 L(t)+\widetilde{C}_5(g\circ\nabla u)(t), \quad t\geq t_0. \tag{4.8}$$

4.1 Case 1: Exponential Decay

If $p=1$, from condition (H1), we know that $\xi(t)g(t)\leq -g'(t)$. Therefore, multiplying both sides of equation (4.8) by $\xi(t)$, and noting that $\xi(t)$ is monotonic and non-increasing $\xi'(t)\leq 0$, we obtain

$$\begin{aligned}(\xi(t)L(t))' &\leq \xi(t)L'(t) \\ &\leq -\widetilde{\epsilon}_0\xi(t)L(t)+\widetilde{C}_5(\xi g\circ\nabla u)(t) \\ &\leq -\widetilde{\epsilon}_0\xi(t)L(t)+\widetilde{C}_5(-g'\circ\nabla u)(t) \\ &\leq -\widetilde{\epsilon}_0\xi(t)L(t)-2\widetilde{C}_5E'(t), \quad t\geq t_0. \end{aligned} \tag{4.9}$$

Let $R(t)=\xi(t)L(t)+2\tilde{C}_5E(t)$. It is easy to see that $R(t)\sim E(t)\sim L(t)$. Therefore, from (4.9), there exists a constant $\epsilon_1>0$ such that

$$R'(t)\leq -\epsilon_1\xi(t)R(t),\ t\geq t_0.$$

By solving the above differential inequality and noting that $R(t)\sim E(t)$. Thus, the exponential decay result in Equation (2.2) is established.

4.2 Case 2: Polynomial Decay

If $1<p<2$, Since $E(t)$ is integrable (as proven above in equations (4.5) and (4.6)), we know that $\|\nabla u(t)\|^2$ is also integrable. Again, by applying Hölder inequality, we obtain

$$\begin{aligned}&\int_0^t g(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|_2^2 ds \\ \leq & \left(\int_0^t \|\nabla u(s)-\nabla u(t)\|_2^2 ds\right)^{\frac{p}{p-1}}\left(\int_0^t g^p(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|_2^2 ds\right)^{\frac{1}{p}} \\ \leq & \left(2\int_0^t\left(\|\nabla u(s))\|_2^2+\|\nabla u(t)\|_2^2\right)ds\right)^{\frac{p}{p-1}}\left(\int_0^t g^p(t-s)\left\|\nabla u(s)-\nabla u(t)\right\|_2^2 ds\right)^{\frac{1}{p}} \\ \leq & C\left((g^p\circ\nabla u)(t)\right)^{\frac{1}{p}}.\end{aligned}$$

Additionally, by Lemma 3.4, we know that

$$L(t)\sim E(t)\sim \|u_t\|_2^2+\|\nabla u\|_2^2+(g\circ\nabla u)(t)+\int_0^1\int_\Omega u_t^2(x,t-\eta\tau(t))\,dx\,d\eta.$$

Therefore, inequality (4.8) becomes: for some $\tilde{\epsilon}_0, C_6>0$,

$$L'(t) \leq -\widetilde{\epsilon}_0 L(t) + C_6 \left((g^p \circ \nabla u)(t) \right)^{\frac{1}{p}}, \quad t \geq t_0$$

holds.

Multiplying both sides of the above equation by $L^{p-1}(t)$ and applying Young's inequality, we can transform the above inequality as follows to simplify energy estimation.

$$\begin{aligned} \frac{1}{p}\left(L^p(t)\right)' &\leq -\widetilde{\epsilon}_0 L^p(t) + C_6 L^{p-1}(t)\left((g^p \circ \nabla u)(t)\right)^{\frac{1}{p}} \\ &\leq -\widetilde{\epsilon}_0 L^p(t) + \frac{\widetilde{\epsilon}_0}{2} L^p(t) + C(\widetilde{\epsilon}_0)\left(g^p \circ \nabla u\right)(t) \\ &\leq -\frac{\widetilde{\epsilon}_0}{2} L^p(t) + C(\widetilde{\epsilon}_0)\left(g^p \circ \nabla u\right)(t), \quad t \geq t_0. \end{aligned}$$

Imilar to Case 1, multiplying both sides by $\xi(t)$, we obtain

$$\begin{aligned} \left(\xi(t)L^p(t)\right)' &\leq -\frac{\widetilde{\epsilon}_0}{2} p\xi(t)L^p(t) + pC(\widetilde{\epsilon}_0)\left(\xi g^p \circ \nabla u\right)(t) \\ &\leq -\frac{\widetilde{\epsilon}_0}{2} p\xi(t)L^p(t) + pC(\widetilde{\epsilon}_0)\left(-g' \circ \nabla u\right)(t) \\ &\leq -\frac{\widetilde{\epsilon}_0}{2} p\xi(t)L^p(t) - 2pC(\widetilde{\epsilon}_0)E'(t), \quad t \geq t_0. \end{aligned}$$

Let $\tilde{R}(t) = \xi(t)L^p(t) + 2pC(\tilde{\epsilon}_0)E(t)$, then $\tilde{R}(t) \sim L(t) \sim E(t)$. Therefore, there exists a constant $\epsilon_2 > 0$ such that

$$\widetilde{R}'(t) \leq -\epsilon_2 \xi(t) \widetilde{R}^p(t), \quad t \geq t_0.$$

By solving the above differential inequality and noting that $R(t) \sim E(t)$. Thus, the polynomial decay result in Equation (2.3) is established.

5. Application

The relaxation function of the viscoelastic damping term plays a key role in the modeling and analysis of viscoelastic materials. It describes how the material exhibits different stress and strain characteristics over time under external forces, particularly in terms of time delay and energy dissipation behavior. By properly modeling and utilizing the relaxation function, we can better understand and predict how viscoelastic materials respond to various mechanical stimuli in real-world engineering applications, and effectively control and optimize the system's dynamic response and damping properties. Future research could explore nonlinear relaxation functions or coupled systems. In this section, we apply the conclusion of Theorem 2.4 to provide specific decay rate forms for the three relaxation function examples given in section 2. These decay estimates can be applied to design materials with optimized energy dissipation properties. In practical applications, we can set the corresponding parameters based on the material's inherent properties to better predict its energy dissipation behavior.

For the kernel function in Example 1, we have the system energy $E(t)$ exhibits an exponential decay rate.

Proof. According to the conditions of the relaxation function (H1), we have

$$1 - \int_0^\infty l e^{-t}\, dt = 1 - (1 - l) = l > 0, \quad g'(t) = -g(t).$$

According to equation (4.8), we have

$$L'(t) \leq -\widetilde{\epsilon}_0 L(t) + \widetilde{C}_5 \left(g \circ \nabla u\right)(t),$$

and

$$\begin{aligned} \left(\xi(t)L(t)\right)' &\leq \xi(t)L'(t) \\ &\leq -\widetilde{\epsilon}_0 \xi(t)L(t) + \widetilde{C}_5\left(\xi g \circ \nabla u\right)(t) \\ &\leq -\widetilde{\epsilon}_0 \xi(t)L(t) + \widetilde{C}_5\left(-g' \circ \nabla u\right)(t) \\ &\leq -\widetilde{\epsilon}_0 \xi(t)L(t) - 2\widetilde{C}_5 E'(t). \end{aligned}$$

Therefore,

$$\left[\xi(t)L(t) + 2\widetilde{C}_5 E'(t)\right]' \leq -\widetilde{\epsilon}_0 \xi(t)L(t)$$

and

$$R'(t) \leqslant -C_7\xi(t)R(t).$$

Integrating the function $R(t)$, we obtain

$$\begin{aligned} \int_0^t \frac{R'(s)}{R(s)}\,ds &\leq \int_0^t -C_7\,ds \\ R(t) &\leq R(0)e^{-C_7 t}. \end{aligned}$$

After simple calculus operations, we can conclude that, for the kernel function in Example 1, the system energy exhibits an exponential decay rate.

From Theorem 2.4 we can conclude that the energy $E(t)$ have a polynomial decay rate for the kernel function in Example 2.

Proof. According to the conditions of the relaxation function (H1), we have

$$\begin{aligned} &g(0) = (1-l)l > 0, 1 - \int_0^\infty \frac{(1-l)l}{(t+1)^{l+1}}\,dt = 1-(1-l) = l > 0, \\ &g'(t) = -\frac{1+l}{t+1}g(t). \end{aligned}$$

Therefore, we know that $\xi(t) = \frac{1+l}{t+1}$. Based on equation (4.8) and Example 1, we have

$$R'(t) \leq -C_8\frac{1+l}{t+1}R(t).$$

By integrating and calculating the function $R(t)$ over the interval from *0* to t, we have

$$\begin{aligned} \int_0^t \frac{R'(s)}{R(s)}\,ds &\leq \int_0^t -\frac{(1+l)C_8}{s+1}\,ds \\ R(t) &\leq R(0)(t+1)^{-(1+l)C_8}. \end{aligned}$$

After simple calculus operations, we can conclude that the kernel function in Example 2 leads to a polynomial decay rate for the system energy.

According to Theorem 2.4, we proceed to prove and derive the uniform decay result in (2.3) for the kernel function in Example 3.

Proof. According to the conditions of the relaxation function (H1), we have

$$\begin{aligned} &g(0) = C_{10} > 0, \quad 1 - \int_0^\infty C_{10}e^{-kt^{p-1}}\,dt = 1-(1-l) = l > 0, \\ &g'(t) \leq -\frac{C_{10}k(p-1)}{C_{10}^p}t^{p-2}g^p(t). \end{aligned}$$

According to equation (4.8), we have

$$\begin{aligned} L'(t) &\leq -\widetilde{\epsilon}_0 L(t) + C_6\left(\left(g^p \circ \nabla u\right)(t)\right)^{\frac{1}{p}} \\ \frac{1}{p}\left(L^p(t)\right)' &\leq -\frac{\widetilde{\epsilon}_0}{2}L^p(t) + C_{12}\left(\left(g^p \circ \nabla u\right)(t)\right), \\ \left(\xi(t)L^p(t)\right)' &\leq -\frac{\widetilde{\epsilon}_0}{2}p\xi(t)L^p(t) - 2pC(\widetilde{\epsilon}_0)E'(t) \\ \left[\xi(t)L^p(t) + 2pC(\widetilde{\epsilon}_0)E(t)\right]' &\leq -\frac{\widetilde{\epsilon}_0}{2}p\xi(t)L^p(t). \end{aligned}$$

Let $\tilde{R}(t) = \xi(t)L^p(t) + 2pC(\tilde{\epsilon}_0)E(t)$. Then, $\tilde{R}(t) \sim L(t) \sim E(t)$, we obtain

$$
\begin{aligned}
\int_0^t \frac{\widetilde{R}'(s)}{\widetilde{R}^p(s)}\,ds &\le \int_0^t -pC_{13}\xi(s)\,ds \\
-\frac{1}{(p-1)\widetilde{R}^{p-1}(t)} + \frac{1}{(p-1)\widetilde{R}^{p-1}(0)} &\le \int_0^t -pC_{13}\xi(s)\,ds \\
\frac{1}{(p-1)\widetilde{R}^{p-1}(0)} + \int_0^t pC_{13}\xi(s)\,ds &\le \frac{1}{(p-1)\widetilde{R}^{p-1}(t)} \\
\widetilde{R}^{p-1}(t) &\le \frac{1}{\frac{1}{(p-1)\widetilde{R}^{p-1}(0)} + \int_0^t pC_{13}\xi(s)\,ds} \\
\widetilde{R}(t) &\le \frac{1}{\left(\frac{1}{\widetilde{R}^{p-1}(0)} + (p-1)\int_0^t pC_{13}\xi(s)\,ds\right)^{\frac{1}{p-1}}} \\
&\le \frac{1}{\left(\frac{1}{\widetilde{R}^{p-1}(0)} + \int_0^t \frac{C_{10}C_{14}k(p-1)}{C_{10}^p} s^{p-2}\,ds\right)^{\frac{1}{p-1}}}.
\end{aligned}
$$

Thus, we have derived the polynomial decay result for the system's energy, as predicted by our earlier analysis, and this completes the proof.

Acknowledgement

This research was supported by the Natural Science Foundation of Guangxi Province (No. 2023GXNSFAA026172).

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