

Superposition of Wave Functions in the G-Qubit Theory

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Abstract

Quantum computing rests upon two theoretical pillars: superposition and entanglement. But some physicists say that this is a very shaky foundation and quantum computing success will have to be based on a different theoretical foundation. The g-qubit theory supports that point of view. Current article is the first one of two and about the superposition principle. Quantum superposition principle states that any two quantum wave functions can be superposed, and the result be another valid wave function. It specifically refers to linearity of the Schrodinger equation. In the g-qubit theory quantum wave functions are identified by points on the surface of three-dimensional sphere S^3 . That gives different, more physically feasible, not mysterious, explanation of what the superposition is.

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1. Introduction. Arithmetic of the G_3^+ elements

For two arbitrary elements of G_3^+ , $g_1 = \alpha_1 + I_{S_1}\beta_1$ and $g_2 = \alpha_2 + I_{S_2}\beta_2$, see [1], with generally different bivector planes of I_{S_1} and I_{S_2} , both I_{S_1} and I_{S_2} assumed to be unit value ones, we have:

$$g_1 + g_2 = (\alpha_1 + \alpha_2) + (I_{S_1}\beta_1 + I_{S_2}\beta_2)$$

The sum of two bivectors $I_{S_1}\beta_1 + I_{S_2}\beta_2$ is a bivector lying in plane different from both I_{S_1} and I_{S_2} (see Figure 1.1)

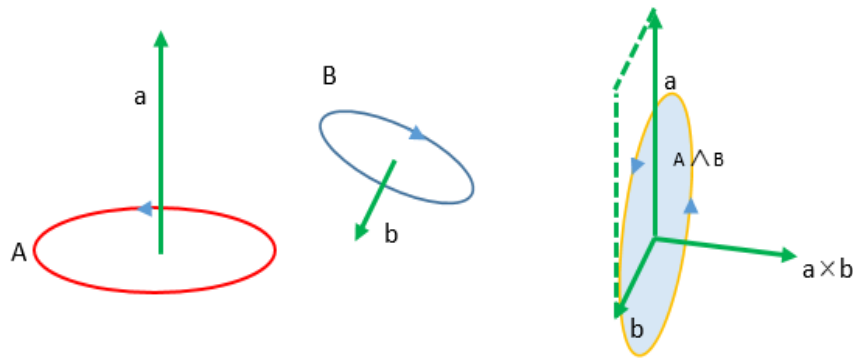


Figure 1.1. Addition of two bivectors using corresponding normal vectors in the right-screw oriented space

To make the result more convenient for calculations expand bivectors I_{S_1} and I_{S_2} in a basis $\{B_1, B_2, B_3\}$ - an arbitrary triple of unit value mutually orthogonal bivectors in three dimensions satisfying, with not critical assumption of right-hand screw orientation $B_1B_2B_3 = 1$, the multiplication rules (see Figure 1.2):

$$B_1B_2 = -B_3, B_1B_3 = B_2, B_2B_3 = -B_1 \text{ (Note 1)}$$

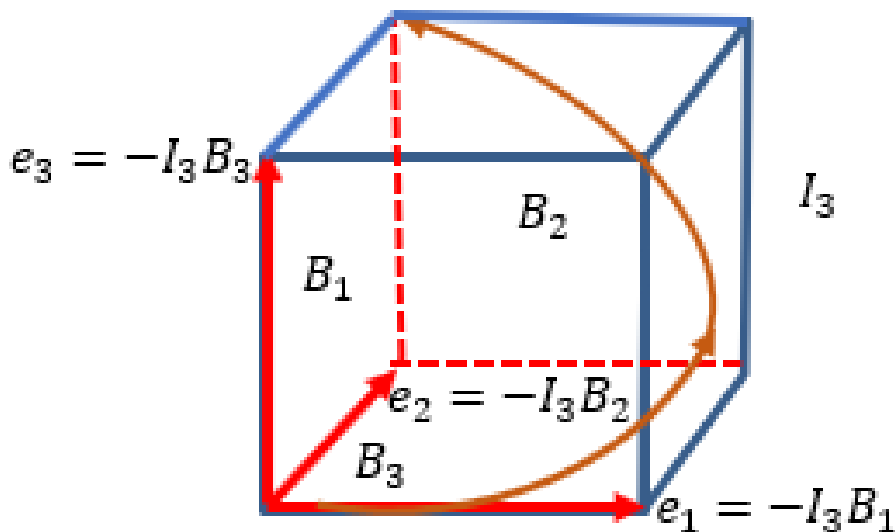


Figure 1.2. Basis of bivectors, vectors associated with the bivectors, and unit value pseudoscalar

Thus:

$$I_{S_1} = b_1^1 B_1 + b_1^2 B_2 + b_1^3 B_3$$

$$I_{S_2} = b_2^1 B_1 + b_2^2 B_2 + b_2^3 B_3$$

Then:

$$I_{S_1} \beta_1 + I_{S_2} \beta_2 = \beta_1 (b_1^1 B_1 + b_1^2 B_2 + b_1^3 B_3) + \beta_2 (b_2^1 B_1 + b_2^2 B_2 + b_2^3 B_3)$$

$$= (\beta_1 b_1^1 + \beta_2 b_2^1) B_1 + (\beta_1 b_1^2 + \beta_2 b_2^2) B_2 + (\beta_1 b_1^3 + \beta_2 b_2^3) B_3$$

and finally

$$g_1 + g_2 = (\alpha_1 + \alpha_2) + (\beta_1 b_1^1 + \beta_2 b_2^1) B_1 + (\beta_1 b_1^2 + \beta_2 b_2^2) B_2 + (\beta_1 b_1^3 + \beta_2 b_2^3) B_3$$

The length of the sum, due to the unit values of I_{S_1} and I_{S_2} , that's $(b_1^1)^2 + (b_1^2)^2 + (b_1^3)^2 = 1$ and $(b_2^1)^2 + (b_2^2)^2 + (b_2^3)^2 = 1$, can be written as

$$\sqrt{(\alpha_1 + \alpha_2)^2 + \beta_1^2 + \beta_2^2 + 2\beta_1\beta_2(s_1 \cdot s_2)}$$

where $s_1 = b_1^1 e_1 + b_1^2 e_2 + b_1^3 e_3$ and $s_2 = b_2^1 e_1 + b_2^2 e_2 + b_2^3 e_3$ (Note 2) are vectors dual to I_{S_1} and I_{S_2} , $s_1 = -I_3 I_{S_1}$, $s_2 = -I_3 I_{S_2}$.

Take an arbitrary $g = \alpha + I_S \beta$. Rewrite it in the form $g = \sqrt{\alpha^2 + \beta^2} \left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} + \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} I_S \right)$. Since $\left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \right)^2 + \left(\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right)^2 = 1$ we can define $\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \equiv \cos \varphi$ and $\frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \equiv \sin \varphi$. In that way we can write $g = \sqrt{\alpha^2 + \beta^2} (\cos \varphi + \sin \varphi I_S)$.

Expand $I_S = b_1 B_1 + b_2 B_2 + b_3 B_3$. Directional cosines $\{b_1, b_2, b_3\}$ define orientation of unit radius disc I_S in three-dimensional space.

Consider sphere of radius $\sqrt{\alpha^2 + \beta^2}$ and intersect it by a plane parallel to I_S . Choose some two-dimensional coordinate system $\{x, y\}$ in that plane. In this way g is fully identified by the sphere of radius $\sqrt{\alpha^2 + \beta^2}$, vector s of the length equal to the sphere radius $\sqrt{\alpha^2 + \beta^2}$ with directional cosines $\{b_1, b_2, b_3\}$, and angle of rotation around s in the coordinate system $\{x, y\}$: $\varphi = \cos^{-1} \left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \right)$ (see Fig.1.3.)

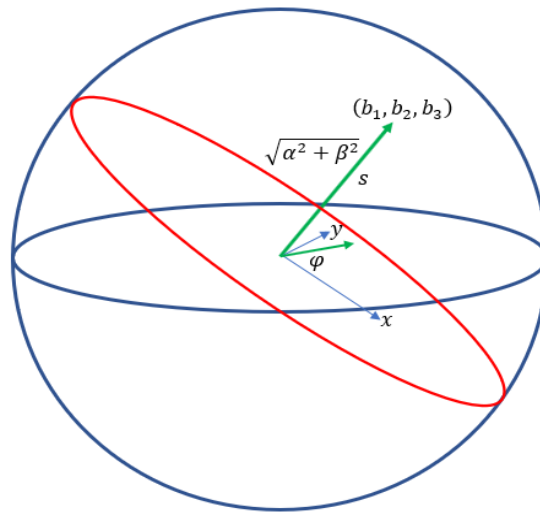


Figure 1.3. Geometric representation of a G_3^+ element

The sum of two G_3^+ elements with vectors s_1 and s_2 associated with unit value bivectors I_{S_1} and I_{S_2} has the associated vector $s = s_1 + s_2$, (see Fig.1.4)

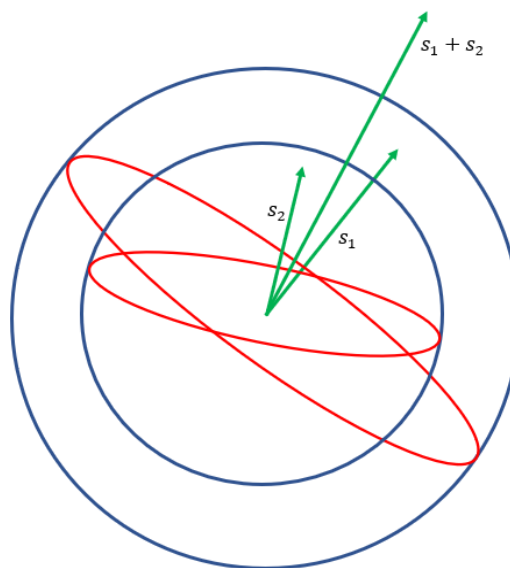


Figure 1.4. Geometric representation of addition

Torsion angle in plane orthogonal to $s_1 + s_2$ is $\cos^{-1}\left(\frac{\alpha_1 + \alpha_2}{\sqrt{(\alpha_1 + \alpha_2)^2 + \beta_1^2 + \beta_2^2 + 2\beta_1\beta_2(s_1 \cdot s_2)}}\right)$.

2. Superposition of Two Basic Wave Functions Corresponding to $|0\rangle$ and $|1\rangle$

The quantum mechanical qubit state, $|\psi\rangle = z_1|0\rangle + z_2|1\rangle$, is linear combination of two basis states $|0\rangle$ and $|1\rangle$. In more details:

$$|\psi\rangle = \begin{pmatrix} \alpha + i\beta_1 \\ \beta_3 + i\beta_2 \end{pmatrix}$$

There exist infinite number of options to select triple $\{B_1, B_2, B_3\}$. Thus, the procedure of recovering a g-qubit associated with $|\psi\rangle = z_1|0\rangle + z_2|1\rangle$ is the following one:

It is necessary, see [2], [3], firstly, to define bivector B_{i_1} in three dimensions identifying the torsion plane. Secondly, choose another bivector B_{i_2} orthogonal to B_{i_1} . The third bivector B_{i_3} , orthogonal to both B_{i_1} and B_{i_2} , is then defined by the first two by orientation (handedness, right screw in the used case): $I_3 B_{i_1} I_3 B_{i_2} I_3 B_{i_3} = I_3$.

Wave functions in the suggested theory are operators acting through measurements on observables:

$$(\overline{\alpha + I_S \beta})C(\alpha + I_S \beta)$$

For any wave function $\alpha + B_{i_1} \beta_i$, $i = 1, 2, 3$, corresponding to $|0\rangle$ (assuming $\alpha^2 + \beta_i^2 = 1$) we get:

$$(\alpha - B_{i_1} \beta_i)B_{i_1}(\alpha + B_{i_1} \beta_i) = (\alpha^2 + \beta_i^2)B_{i_1} = B_{i_1}$$

For the wave functions $\beta_{(i+2) \bmod 3} B_{(i+2) \bmod 3} + \beta_{(i+1) \bmod 3} B_{(i+1) \bmod 3}$, $i = 1, 2, 3$, corresponding to $|1\rangle$ (with the agreement $3 \bmod 3 = 3$) the value of observable B_{i_1} is (with same assumption $\beta_{(i+2) \bmod 3}^2 + \beta_{(i+1) \bmod 3}^2 = 1$):

$$\begin{aligned} &(-\beta_{(i+2) \bmod 3} B_{(i+2) \bmod 3} - \beta_{(i+1) \bmod 3} B_{(i+1) \bmod 3})B_{i_1}(\beta_{(i+2) \bmod 3} B_{(i+2) \bmod 3} + \beta_{(i+1) \bmod 3} B_{(i+1) \bmod 3}) \\ &= -(\beta_{(i+2) \bmod 3}^2 + \beta_{(i+1) \bmod 3}^2)B_{i_1} = -B_{i_1} \end{aligned}$$

Let us take an arbitrary bivector observable expanded in basis $\{B_1, B_2, B_3\} \equiv \{B_{i_1}, B_{i_2}, B_{i_3}\}$:

$$C = C_1 B_1 + C_2 B_2 + C_3 B_3$$

The result of measurement by wave function corresponding to $|0\rangle$ is:

$$\begin{aligned} (\alpha - \beta_1 B_1)C(\alpha + \beta_1 B_1) &= C_1 B_1 + [C_2(\alpha^2 - \beta_1^2) - 2C_3 \alpha \beta_1]B_2 + [C_3(\alpha^2 - \beta_1^2) + 2C_2 \alpha \beta_1]B_3 = C_1 B_1 + \\ &(C_2 \cos 2\varphi - C_3 \sin 2\varphi)B_2 + (C_2 \sin 2\varphi + C_3 \cos 2\varphi)B_3, \end{aligned} \tag{2.1}$$

using parametrization $\alpha = \cos \varphi$, $\beta_1 = \sin \varphi$.

The result of measurement by wave function corresponding to $|1\rangle$ is:

$$\begin{aligned} (-\beta_2 B_2 - \beta_3 B_3)C(\beta_2 B_2 + \beta_3 B_3) &= -C_1 B_1 + [C_2(\beta_2^2 - \beta_3^2) + 2C_3 \beta_2 \beta_3]B_2 + [2C_2 \beta_2 \beta_3 - C_3(\beta_2^2 - \beta_3^2)]B_3 = \\ &-C_1 B_1 + (C_2 \cos 2\theta + C_3 \sin 2\theta)B_2 + (C_2 \sin 2\theta - C_3 \cos 2\theta)B_3, \end{aligned} \tag{2.2}$$

with $\beta_2 = \cos \theta$, $\beta_3 = \sin \theta$.

This is a deeper result compared with conventional quantum mechanics where

$$|0\rangle = \begin{pmatrix} \alpha + i\beta_1 \\ 0 \end{pmatrix} \text{ and } |1\rangle = \begin{pmatrix} 0 \\ \beta_3 + i\beta_2 \end{pmatrix}$$

Conclusion:

- Measurement of observable $C = C_1 B_1 + C_2 B_2 + C_3 B_3$ by any wave function corresponding to $|0\rangle$ is bivector with the B_1 component equal to unchanged value C_1 . The B_2 and B_3 components of the result of measurement are equal to B_2 and B_3 components of C rotated by angle 2φ defined by $\alpha = \cos \varphi$, $\beta_1 = \sin \varphi$ where plane of rotation is B_1 .
- Measurement of observable $C = C_1 B_1 + C_2 B_2 + C_3 B_3$ by any wave function corresponding to $|1\rangle$ is bivector with the B_1 component equal to flipped value $-C_1$. The B_2 and B_3 components of the result of measurement are equal to B_2 and B_3 components of C rotated by angle 2θ defined by $\beta_2 = \cos \theta$, $\beta_3 = \sin \theta$ where plane of rotation is B_1 but direction of rotation is opposite to the case of $|0\rangle$.

If we denote by $so_{|0\rangle}$ and $so_{|1\rangle}$ arbitrary wave functions represented in G_3^+ by $\alpha + \beta_1 B_1$ and $\beta_2 B_2 + \beta_3 B_3 = (\beta_3 + \beta_2 B_1)B_3$ they only differ by factor B_3 in $so_{|1\rangle}$, thus for the measurement by them we have:

$$\overline{so_{|1\rangle}} C so_{|1\rangle} = \overline{B_3 so_{|0\rangle}} C so_{|0\rangle} B_3$$

That simply means that the measurement on the left side is received from $\overline{so_{|0\rangle}} C so_{|0\rangle}$ by its flipping in plane B_3 (see Figure 2.1)

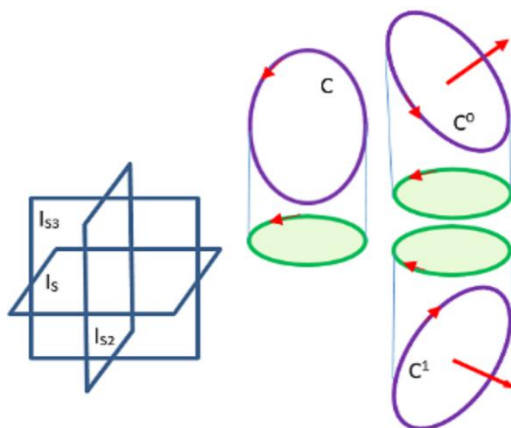


Figure 2.1. Difference in measuring by $s_{0|0}$ and $s_{0|1}$

Probabilities of the results of measurements are measures of wave functions on S^3 surface giving considered results.

Suppose we are interested in the probability of the result of measurement in which the observable component

C_1B_1 does not change. This is relative measure of wave functions $\sqrt{\alpha^2 + \beta_1^2} \left(\frac{\alpha}{\sqrt{\alpha^2 + \beta_1^2}} + \frac{\beta_1}{\sqrt{\alpha^2 + \beta_1^2}} B_1 \right)$ in the

measurements:

$$\sqrt{\alpha^2 + \beta_1^2} \left(\frac{\alpha}{\sqrt{\alpha^2 + \beta_1^2}} - \frac{\beta_1}{\sqrt{\alpha^2 + \beta_1^2}} B_1 \right) C \sqrt{\alpha^2 + \beta_1^2} \left(\frac{\alpha}{\sqrt{\alpha^2 + \beta_1^2}} + \frac{\beta_1}{\sqrt{\alpha^2 + \beta_1^2}} B_1 \right) \tag{2.3}$$

That measure is equal to $\alpha^2 + \beta_1^2$, that is equal to z_1^2 in the down mapping from G_3^+ to Hilbert space of $z_1|0\rangle + z_2|1\rangle$. Thus, we have clear explanation of common quantum mechanics wisdom on “probability of finding system in state $|0\rangle$ ”.

Similar calculations explain correspondence of $\beta_2^2 + \beta_3^2$ to z_2^2 in the qubit $z_1|0\rangle + z_2|1\rangle$ when the component C_1B_1 in measurement just got flipped.

Let us consider superposition of $\alpha + \beta_1B_1$ and $\beta_2B_2 + \beta_3B_3$ with some coefficients p_1 and p_2 (Note 3),

$$p_1(\alpha + \beta_1B_1) + p_2(\beta_2B_2 + \beta_3B_3),$$

and measuring by it of $C = C_1B_1 + C_2B_2 + C_3B_3$.

$$\begin{aligned} & [p_1(\alpha - \beta_1B_1) + p_2(-\beta_2B_2 - \beta_3B_3)]C[p_1(\alpha + \beta_1B_1) + p_2(\beta_2B_2 + \beta_3B_3)] \\ &= p_1(\alpha - \beta_1B_1)Cp_1(\alpha + \beta_1B_1) + p_2(-\beta_2B_2 - \beta_3B_3)Cp_2(\beta_2B_2 + \beta_3B_3) \\ &+ p_2(-\beta_2B_2 - \beta_3B_3)Cp_1(\alpha + \beta_1B_1) + p_1(\alpha - \beta_1B_1)Cp_2(\beta_2B_2 + \beta_3B_3) \\ &= p_1(\alpha - \beta_1B_1)Cp_1(\alpha + \beta_1B_1) + p_2(-\beta_2B_2 - \beta_3B_3)Cp_2(\beta_2B_2 + \beta_3B_3) \\ &+ p_1(\alpha - \beta_1B_1)Cp_1(\alpha + \beta_1B_1)p_1(\alpha - \beta_1B_1)p_2(\beta_2B_2 + \beta_3B_3) \\ &+ p_2(-\beta_2B_2 - \beta_3B_3)Cp_2(\beta_2B_2 + \beta_3B_3)p_2(-\beta_2B_2 - \beta_3B_3)p_1(\alpha + \beta_1B_1) \end{aligned}$$

It follows from this formula that the result of measurement by wave function $p_1(\alpha + \beta_1B_1) + p_2(\beta_2B_2 + \beta_3B_3)$ makes the C_1B_1 component unchanged and two other components rotated around the normal to B_1 , see (2.1) and (2.3), with probability p_1^2 (item $p_1(\alpha - \beta_1B_1)Cp_1(\alpha + \beta_1B_1)$). Then it just flips the C_1B_1 component and two other components rotated around the normal to B_1 , but in opposite direction see (2.2) with probability p_2^2 (item $p_2(-\beta_2B_2 - \beta_3B_3)Cp_2(\beta_2B_2 + \beta_3B_3)$). Other two items are correspondingly the first above result subjected to

Clifford (parallel) translation on \mathbb{S}^3 by $p_1 p_2 (\alpha - \beta_1 B_1) (\beta_2 B_2 + \beta_3 B_3)$ and the second result subjected to opposite Clifford translation $p_1 p_2 (-\beta_2 B_2 - \beta_3 B_3) (\alpha + \beta_1 B_1)$. They are neither (2.1) nor (2.2) and their probabilities to make $C_1 B_1$ unchanged or flipped are zero. Though, they give two other different available measurement results.

3. Superposition of two arbitrary wave functions

Any arbitrary G_3^+ wave function $\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$ can be rewritten either as 0-type wave function or 1-type wave function:

$$\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 = \alpha + I_{S(\beta_1, \beta_2, \beta_3)} \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2},$$

$$\text{where } I_{S(\beta_1, \beta_2, \beta_3)} = \frac{\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}}, \quad \text{0-type,}$$

or

$$\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 = (\beta_3 + \beta_2 B_1 - \beta_1 B_2 - \alpha B_3) B_3 = (\beta_3 + I_{S(\beta_2, -\beta_1, -\alpha)} \sqrt{\alpha^2 + \beta_1^2 + \beta_2^2}) B_3,$$

$$\text{where } I_{S(\beta_2, -\beta_1, -\alpha)} = \frac{\beta_2 B_1 - \beta_1 B_2 - \alpha B_3}{\sqrt{\alpha^2 + \beta_1^2 + \beta_2^2}}, \quad \text{1-type.}$$

All that means that any G_3^+ wave function $\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$ measuring observable $C_1 B_1 + C_2 B_2 + C_3 B_3$ does not change the observable projection onto plane of $I_{S(\beta_1, \beta_2, \beta_3)} = \frac{\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3}{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}}$ and just flips the

observable projection onto plane $I_{S(\beta_2, -\beta_1, -\alpha)} = \frac{\beta_2 B_1 - \beta_1 B_2 - \alpha B_3}{\sqrt{\alpha^2 + \beta_1^2 + \beta_2^2}}$.

Take two arbitrary wave functions and rewrite the first one as 0-type wave function and the second one as 1-type wave function. Then all the results of Sec.2 become applicable for their superposition. It will follow that there will be a result of measurement

$$p_1^2 \left(\alpha - I_{S(\beta_1, \beta_2, \beta_3)} \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} \right) C \left(\alpha + I_{S(\beta_1, \beta_2, \beta_3)} \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} \right)$$

not changing the projection of C onto plane of $I_{S(\beta_1, \beta_2, \beta_3)}$ and keeping probability p_1^2 ; plus, result of measurement

$$p_2^2 \left(-B_3 \left(\beta_3 - I_{S(\beta_2, -\beta_1, -\alpha)} \sqrt{\alpha^2 + \beta_1^2 + \beta_2^2} \right) \right) C \left(\left(\beta_3 + I_{S(\beta_2, -\beta_1, -\alpha)} \sqrt{\alpha^2 + \beta_1^2 + \beta_2^2} \right) B_3 \right)$$

just flipping projection of C in plane of $I_{S(\beta_2, -\beta_1, -\alpha)}$ and keeping probability p_2^2 . Two other results represent the first two subjected to Clifford (parallel) translations on the sphere \mathbb{S}^3 by

$$p_1 p_2 \left(\alpha - I_{S(\beta_1, \beta_2, \beta_3)} \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} \right) \left(\beta_3 + I_{S(\beta_2, -\beta_1, -\alpha)} \sqrt{\alpha^2 + \beta_1^2 + \beta_2^2} \right)$$

and

$$p_1 p_2 \left(\beta_3 - I_{S(\beta_2, -\beta_1, -\alpha)} \sqrt{\alpha^2 + \beta_1^2 + \beta_2^2} \right) \left(\alpha + I_{S(\beta_1, \beta_2, \beta_3)} \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2} \right)$$

correspondingly.

4. Conclusions

Superposition of any two wave functions in the frame of g-qubit theory gives another wave function the result of measurement by which is more complicated than in conventional quantum mechanics. In addition to the two results of measurements coming from composed items of the wave functions there appear two additional items which are Clifford (parallel) translations of the first two results in opposite directions on the sphere \mathbb{S}^3 .

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Notes

Note 1. Opposite orientation $B_1B_2B_3 = -1$ can be equivalently used

Note 2. Length $\sqrt{\alpha^2 + \beta^2}$ of vector s associated with $g = \alpha + I_s\beta$ will be also called *module* of g and denoted $|g|$

Note 3. Due to orthogonality of $\alpha + \beta_1B_1$ and $\beta_2B_2 + \beta_3B_3$ in Euclidean sense on the three-sphere $s_1 \cdot s_2 = 0$ in this case

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